Real Business Cycle

Jin Cao\textsuperscript{1}

\textsuperscript{1}Munich Graduate School of Economics

Ludwig-Maximilians-Universität München (WS07/08)
Outline

Motivation
- Facts of Business Cycles
- Research Question and Roadmap
- Distinguished Features of RBC Models

RBC: Solving a Baseline Model
- Model Specifications
- Characterizing the Dynamic System
- Linear Rational Expectation System

Calibration and Simulation
- Calibration
- Solving for Rational Expectation Equilibrium
- Numerical Simulation
Motivation: Facts of Business Cycles

- Business cycle: Major macro indicators, consumption, labor supply, investment... fluctuating along with the boom and bust of the economic growth. Some fairly robust facts:
  - Output movements in different sectors of the economy exhibit a high degree of coherence;
  - Investment is about three times more volatile than GDP;
  - Consumption is about half as volatile as GDP;
  - Employment is about eighty percent as volatile as GDP;
  - The capital stock is much less variable than output;
  - Consumption-output ratio is strongly counter-cyclical and investment-output ratio is strongly pro-cyclical.

- An instantaneous question: Are we able to explain?
How Far can We Go with What We Leaned?

- What have we (or people in 1980) learned so far?
  - Dynamic? Yes, as we always do!
  - Stochastic? Yes, asset pricing, optimization under uncertainty.
  - General equilibrium? Yes!
    - Partial equilibrium (Solow)
    - General equilibrium (Ramsey-Cass-Koopmans)
    - General equilibrium with labor-leisure choice (Ex. 2-4).

- Then: How many facts can be explained by these theories?
  
  RBC Models:
  - Combine those widely accepted;
  - Do “the reasonable things to do”, go on with the best practices;
  - Prescott: Progress, don’t regress!
What Makes an RBC Model?

- Micro-Based (RCK): Infinitely lived representative agent;

- Dynamic optimization. Employment in the economy is determined by the individual’s labor-leisure choice;

- Driving force of business cycles: Exogenous shocks to technology. Transmission mechanism: Productivity shock → Real interest rate / wage → Decision in labor supply / Consumption → Investment → Future capital stock...

- The markets are complete and there is no asymmetric information. All the agents have rational expectation and the markets clear continuously: The business cycle is REAL!
The representative agent in this economy considers the following dynamic optimization problem

\[
\max \{c_t, l_t, k_t\}_{t=0}^{+\infty} \quad \sum_{t=0}^{+\infty} \beta^t [\theta \ln c_t + (1 - \theta) \ln (1 - l_t)],
\]

\[
s.t. \quad c_t + k_{t+1} = z_t k_t^\alpha l_t^{1-\alpha} + (1 - \delta) k_t,\]

\[
\ln z_{t+1} = \rho \ln z_t + \varepsilon_{t+1}.
\]

Neoclassical production function \( f(k_t, l_t) = k_t^\alpha l_t^{1-\alpha} \) with productivity shock: \( \varepsilon_{t+1} \sim N(0, \sigma^2) \), \( \rho \in (0, 1) \).
The Optimality Conditions

- Define Bellman equation

\[
V(k, z) = \max_{(c, l, k')} \left\{ \theta \ln c + (1 - \theta) \ln(1 - l) + \beta E \left[ V(k', z') \right] \right\},
\]

s.t. \( c + k' = zk^{\alpha}l^{1-\alpha} + (1 - \delta)k. \)

The first order conditions are

\[
c_t = \frac{\theta}{1 - \theta} (1 - l_t)(1 - \alpha)z_t k_t^{\alpha}l_t^{-\alpha},
\]

\[
\frac{1}{c_t} = \beta E_t \left\{ \frac{1}{c_{t+1}} \left[ \alpha z_{t+1} k_{t+1}^{\alpha-1}l_{t+1}^{1-\alpha} + (1 - \delta) \right] \right\},
\]

\[
c_t = z_t k_t^{\alpha}l_t^{1-\alpha} + (1 - \delta)k_t - k_{t+1}.
\]
Benchmark: The Steady State

- The steady state for this economy is featured by
  - The technology shock is constant, $z_t = 1$, $\forall t$;
  - Capital, labor, and consumption are constant, i.e. $k_t = k^*$, $c_t = c^*$, and $l_t = l^*$, $\forall t$.

- Then $\beta$ and $\theta$ can be determined by the other steady state variables.

$$c^* = \frac{\theta}{1 - \theta} \left(1 - l^*\right)\left(1 - \alpha\right)k^*\alpha l^* - \alpha,$$

$$1 = \beta \left[\alpha \frac{y^*}{k^*} + (1 - \delta)\right],$$

$$c^* = y^* - \delta k^*.$$
Log-Linearization and Taylor Expansion

- Difficulty: System is non-linear, hard to say anything about the equilibrium path
  - Does the equilibrium path exist at all?
  - If it exists, how does it look like?

- Remember that we can log-linearize a function $f(X)$ around $X^*$
  
  $$f(X) = f(e^{\ln X}) = f(X^*) + f'(X^*)X^*(\ln X - \ln X^*).$$

- Define $\hat{X} = \ln X - \ln X^* = \ln \left(1 + \frac{X - X^*}{X^*}\right) \approx \frac{X - X^*}{X^*} \quad \hat{X}$ is the percentage deviation from $X^*$! A normalization.
Log-Linearization and Taylor Expansion

Linearized system around the steady state — But then?

\[ \hat{c}_t = - \left( \frac{l^*}{1-l^*} + \alpha \right) \hat{l}_t + \alpha \hat{k}_t + \hat{z}_t, \]

\[ -\hat{c}_t = -E_t[\hat{c}_{t+1}] + \beta \alpha k^* \alpha^{-1} l^* 1^{-\alpha} \left\{ (\alpha - 1) E_t[\hat{k}_{t+1}] + E_t[\hat{z}_{t+1}] + (1 - \alpha) E_t[\hat{l}_{t+1}] \right\}, \]

\[ \hat{k}_{t+1} = (k^* \alpha^{-1} l^* 1^{-\alpha}) \hat{z}_t + \left[ \alpha k^* \alpha^{-1} l^* 1^{-\alpha} + (1 - \delta) \right] \hat{k}_t + (1 - \alpha) k^* \alpha^{-1} l^* 1^{-\alpha} \hat{l}_t - \frac{c^*}{k^*} \hat{c}_t, \]

\[ \rho \hat{z}_t = E_t[\hat{z}_{t+1}]. \]
Separate the realized variables from the expected values — Linear expectation system

\[
\begin{pmatrix}
1 & -\alpha & \frac{l^*}{1-l^*} + \alpha & -1 \\
-1 & 0 & 0 & 0 \\
-\frac{c^*}{k^*} & \alpha k^\alpha - 1 k^\alpha - 1 - \alpha + (1 - \delta) & (1 - \alpha) k^\alpha - 1 k^\alpha - 1 - \alpha & k^\alpha - 1 k^\alpha - 1 - \alpha \\
0 & 0 & 0 & \rho
\end{pmatrix}
\begin{pmatrix}
\hat{c}_t \\
\hat{k}_t \\
\hat{l}_t \\
\hat{z}_t
\end{pmatrix}
= \begin{pmatrix}
0 & 0 & 0 & 0 \\
-1 & \beta \alpha k^\alpha - 1 k^\alpha - 1 - \alpha (\alpha - 1) & \beta \alpha k^\alpha - 1 k^\alpha - 1 - \alpha (1 - \alpha) & \beta \alpha k^\alpha - 1 k^\alpha - 1 - \alpha \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
E_t
\begin{pmatrix}
\hat{c}_{t+1} \\
\hat{k}_{t+1} \\
\hat{l}_{t+1} \\
\hat{z}_{t+1}
\end{pmatrix}.
\]
Finding the Plausible Equilibria

- Now we have already built up the relation between $\hat{x}_t$ and $E_t[\hat{x}_{t+1}]$: $A\hat{x}_t = BE_t[\hat{x}_{t+1}]$. *Jordan decomposition*

$\hat{x}_t = A^{-1}BE_t[\hat{x}_{t+1}]$,

$\hat{x}_t = VDV^{-1}E_t[\hat{x}_{t+1}]$,

$V^{-1}\hat{x}_t = DV^{-1}E_t[\hat{x}_{t+1}]$.

- $D$ is a diagonal matrix whose diagonal elements are the eigenvalues of $A^{-1}B$. Now: Which eigenvalue relates to a sensible equilibrium path? “Solution concept” — REE!
Calibration: Relate the Model to Real World

- The central question still remains: Could we explain the Facts by the model at hand?

- To do this, relate the model to real world data and explicitly solve the linear expectation system!

- Some economic facts: \( \alpha = \frac{1}{3}, \frac{c^*}{y^*} = \frac{2}{3}, l^* = 0.3, \delta = 0.025\) (quarterly), \( \frac{k^*}{y^*} = 11 \). Feed them into the steady state equations and get \( \beta \) and \( \theta \), then calculate \( c^*, k^*, y^* \).

- From US (1946-1986) I got \( c^* = 0.79, k^* = 10.90, l^* = 0.29, y^* = 1.06, \beta = 0.99, \delta = 0.025, \alpha = 0.36, \theta = 0.32, \rho = 0.95 \).
Calibrated Linear Expectation System

 Now feed these numbers into our linear expectation system

\[
\begin{bmatrix}
1 & -0.36 & 0.77 & -1 \\
-1 & 0 & 0 & 0 \\
-0.072 & 1.010 & 0.062 & 0.097 \\
0 & 0 & 0 & 0.95
\end{bmatrix}
\begin{bmatrix}
\hat{c}_t \\
\hat{k}_t \\
\hat{l}_t \\
\hat{z}_t
\end{bmatrix}
= 
\begin{bmatrix}
0 & 0 & 0 & 0 \\
-1 & -0.022 & 0.022 & 0.035 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
E_t
\begin{bmatrix}
\hat{c}_{t+1} \\
\hat{k}_{t+1} \\
\hat{l}_{t+1} \\
\hat{z}_{t+1}
\end{bmatrix},
\]

\[
\begin{bmatrix}
\hat{c}_t \\
\hat{k}_t \\
\hat{l}_t \\
\hat{z}_t
\end{bmatrix}
= 
\begin{bmatrix}
1.000 & 0.0220 & -0.0220 & -0.0350 \\
0.1468 & 0.9657 & -0.0032 & -0.1850 \\
-1.2301 & 0.4229 & 0.0271 & 1.3260 \\
0 & 0 & 0 & 1.0526
\end{bmatrix}
E_t
\begin{bmatrix}
\hat{c}_{t+1} \\
\hat{k}_{t+1} \\
\hat{l}_{t+1} \\
\hat{z}_{t+1}
\end{bmatrix}
\]

 and then apply Jordan decomposition.
Jordan Decomposition

\[
\begin{bmatrix}
0 & 0 & 0 & 46.4423 \\
-1.6329 & -0.2479 & 0.0359 & -45.8191 \\
-1.6611 & 0.9464 & 0.0365 & 0.6271 \\
1.2629 & -0.4547 & 0.9725 & -1.2629 \\
\end{bmatrix}
\begin{bmatrix}
\hat{c}_t \\
\hat{k}_t \\
\hat{l}_t \\
\hat{z}_t \\
\end{bmatrix}
\]

\[
= 
\begin{bmatrix}
1.0526 & 0 & 0 & 0 \\
0 & 1.0493 & 0 & 0 \\
0 & 0 & 0.9434 & 0 \\
0 & 0 & 0 & 0 \\
\end{bmatrix}
\begin{bmatrix}
0 & 0 & 0 & 46.4423 \\
-1.6329 & -0.2479 & 0.0359 & -45.8191 \\
-1.6611 & 0.9464 & 0.0365 & 0.6271 \\
1.2629 & -0.4547 & 0.9725 & -1.2629 \\
\end{bmatrix}
E_t
\begin{bmatrix}
\hat{c}_{t+1} \\
\hat{k}_{t+1} \\
\hat{l}_{t+1} \\
\hat{z}_{t+1} \\
\end{bmatrix}.
\]

- The eigenvalues of $A^{-1}B$ are $\lambda_1 = 1.0526$, $\lambda_2 = 1.0493$, $\lambda_3 = 0.9434$, $\lambda_4 = 0$.

- Question: Which one(s) relate(s) to sensible equilibrium path(s)?
Rational expectation equilibrium (REE): Determinancy out of perfect foresight!

\[
\begin{bmatrix}
\hat{d}_{1,t} \\
\hat{d}_{2,t} \\
\hat{d}_{3,t} \\
\hat{d}_{4,t}
\end{bmatrix} = 
\begin{bmatrix}
1.0526 & 0 & 0 & 0 \\
0 & 1.0493 & 0 & 0 \\
0 & 0 & 0.9434 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
E_t 
\begin{bmatrix}
\hat{d}_{1,t+1} \\
\hat{d}_{2,t+1} \\
\hat{d}_{3,t+1} \\
\hat{d}_{4,t+1}
\end{bmatrix},
\]

\[
\hat{d}_{i,t} = \lim_{j \to +\infty} \lambda_i^j E_t \left[ \hat{d}_{i,t+j} \right].
\]

The only determinancy comes from \( \lambda_3 = 0.9434 \), therefore the only REE is \( \hat{d}_{3,t} = 0 \).
Rational Expectation System

- Now we can write down the complete stable system under rational expectation equilibrium
  - REE: $\hat{d}_{3,t} = 0$
    \[ \hat{c}_t = 0.570\hat{k}_t + 0.022\hat{l}_t + 0.378\hat{z}_t; \]
  - Intratemporal efficiency condition:
    \[ 0 = \hat{c}_t - 0.36\hat{k}_t + 0.77\hat{l}_t - \hat{z}_t. \]
  - Intertemporal efficiency condition:
    \[ \hat{k}_{t+1} = -0.072\hat{c}_t + 1.010\hat{k}_t + 0.062\hat{l}_t + 0.097\hat{z}_t; \]
  - State $(k_t, z_t) \rightarrow$ control $(c_t, l_t) \rightarrow$ next period $k_{t+1}$ ...
Duplicate the Business Cycle under Continuous Shocks

Feed \( \ln z_{t+1} = \rho \ln z_t + \varepsilon_{t+1} \) \((\sigma = 0.007)\) into the linear rational expectation system under MATLAB / Mathematica \((c\text{-green, } y\text{-blue, } i\text{-red})\)
Evolution after One-Time Shock

One-time shock: $\ln z_0 = 0.007$, $\ln z_{t+1} = \ln \rho z_t$, $\forall t \geq 0$
Numerical Result: How Successful We are Now?

- Then I feed 2000 observations in MATLAB and see the precise standard deviations and correlations — RBC does a great job in reproducing the reality!

<table>
<thead>
<tr>
<th>Time Series</th>
<th>Data</th>
<th>Model</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( \sigma_x )</td>
<td>( \text{Corr}(x, y) )</td>
</tr>
<tr>
<td>Output ( y )</td>
<td>1.71</td>
<td>1.00</td>
</tr>
<tr>
<td>Consumption ( c )</td>
<td>0.84</td>
<td>0.76</td>
</tr>
<tr>
<td>Investment ( i )</td>
<td>5.38</td>
<td>0.90</td>
</tr>
<tr>
<td>Capital ( k )</td>
<td>0.62</td>
<td>(-0.08)</td>
</tr>
<tr>
<td>Employment ( l )</td>
<td>1.65</td>
<td>0.88</td>
</tr>
</tbody>
</table>
Summary: Solved Problems

- What have we done so far?
  - A genuine RBC model adapted from Kydland & Prescott (1982), and Hansen (1985);
  - Solution methods proposed by Blanchard & Kahn (1980), Sims (1980), Campbell (1994), Uhlig (1997);

- What may make us satisfied so far?
  - A remarkable job on accounting for main business cycle facts;
  - Substantial shift in interpretation of business cycles: fluctuations as efficient adjustment to productivity, rather than temporary breakdowns of the market mechanism!
  - Therefore, “First Best Solution”!
Summary: Critiques and Unsettled Issues

- **Open questions:**
  - No role for government! No justification for government intervention.
  - Where do the productivity shocks come from?

- **Extensions and Current Research:**
  - Genuine RBC research is already inactive;
  - Many extensions since then: Labor market imperfections, government expenditure shocks, "home production", variable capital utilization...
  - Many active studies based on RBC: Monetary models (!), RBC in open economy, heterogenous agents, DSGE econometrics ...
For Further Reading... 1

Stadler, G. W.  
*Real Business Cycles.*  

Cooley, T. F. (Editor)  
*Frontiers of Business Cycle Research.*  