First we solve the standard Ramsey model as a baseline case.

1 Standard Ramsey Model

Rewrite the expression of $\frac{\dot{\hat{c}}(t)}{\hat{c}(t)}$ into log-linearized form

$$\frac{d\ln\hat{c}}{dt} = \alpha\exp\left[(\alpha-1)\ln\hat{k}\right] - \delta - \rho,$$

as well as the expression of $\frac{\dot{\hat{k}}(t)}{\hat{k}(t)}$

$$\frac{d\ln\hat{k}}{dt} = \exp\left[(\alpha-1)\ln\hat{k}\right] - \exp\left[\ln\left(\frac{\hat{c}}{\hat{k}}\right)\right] - \delta.$$

Applying first order Taylor expansion to these two equations around steady state, it's simple to get $\frac{\dot{\hat{c}}}{\hat{c}}$

$$\begin{aligned}\frac{d\ln\hat{c}}{dt} &= \alpha(\alpha-1)\exp\left[(\alpha-1)\ln\hat{k}^*\right]\left[\ln\left(\frac{\hat{k}}{\hat{k}^*}\right)\right] \\ &= (\alpha-1)(\rho+\delta)\left[\ln\left(\frac{\hat{k}}{\hat{k}^*}\right)\right],\end{aligned}$$

as well as $\frac{\dot{\hat{k}}}{\hat{k}}$

$$\begin{aligned}\frac{d\ln\hat{k}}{dt} &= \left\{(\alpha-1)\exp\left[(\alpha-1)\ln\hat{k}^*\right] + \exp\left[\ln\left(\frac{\hat{c}^*}{\hat{k}^*}\right)\right]\right\}\left[\ln\left(\frac{\hat{k}}{\hat{k}^*}\right)\right] \\ &\quad - \exp\left[\ln\left(\frac{\hat{c}^*}{\hat{k}^*}\right)\right]\left[\ln\left(\frac{\hat{c}}{\hat{c}^*}\right)\right] \\ &= \rho\left[\ln\left(\frac{\hat{k}}{\hat{k}^*}\right)\right] - \left(\frac{\rho+\delta}{\alpha} - \delta\right)\left[\ln\left(\frac{\hat{c}}{\hat{c}^*}\right)\right].\end{aligned}$$

For simplicity, rewrite the linearized system as

$$\begin{bmatrix} \frac{d\ln\hat{k}}{dt} \\ \frac{d\ln\hat{c}}{dt} \end{bmatrix} = \begin{bmatrix} \rho & -\left(\frac{\rho+\delta}{\alpha} - \delta\right) \\ (\alpha-1)(\rho+\delta) & 0 \end{bmatrix}\begin{bmatrix} \ln\left(\frac{\hat{k}}{\hat{k}^*}\right) \\ \ln\left(\frac{\hat{c}}{\hat{c}^*}\right) \end{bmatrix} = \mathbf{A}\begin{bmatrix} \ln\left(\frac{\hat{k}}{\hat{k}^*}\right) \\ \ln\left(\frac{\hat{c}}{\hat{c}^*}\right) \end{bmatrix}. \tag{1}$$

To find the eigenvalues of matrix $\mathbf{A}$, solve

$$\begin{vmatrix} \rho-\lambda & -\left(\frac{\rho+\delta}{\alpha} - \delta\right) \\ (\alpha-1)(\rho+\delta) & -\lambda \end{vmatrix} = 0 \tag{2}$$

for λ and this gives

$$\lambda_{1R} = \frac{\rho - \sqrt{\rho^2 + 4(1-\alpha)(\rho+\delta)\left(\frac{\rho+\delta}{\alpha} - \delta\right)}}{2} < 0, \tag{3}$$

$$\lambda_{2R} = \frac{\rho + \sqrt{\rho^2 + 4(1-\alpha)(\rho+\delta)\left(\frac{\rho+\delta}{\alpha} - \delta\right)}}{2} > 0, \tag{4}$$

showing the existence of saddle path (λ_{1R} is the stable solution), and the subscript R denotes standard Ramsey model.

2 Ramsey Model with Labor / Leisure Choice

Rewrite the dynamic system using the result of question **c)**

$$\begin{aligned}
\frac{\dot{\hat{c}}(t)}{\hat{c}(t)} &= \alpha\left[\hat{k}(t)\right]^{\alpha-1} - \delta - \rho - \left(-\frac{1}{1+\eta}\frac{\dot{\hat{c}}(t)}{\hat{c}(t)} + \frac{\alpha}{1+\eta}\frac{\dot{\hat{k}}(t)}{\hat{k}(t)}\right), \\
\frac{\dot{\hat{k}}(t)}{\hat{k}(t)} &= \hat{k}(t)^{\alpha-1} - \frac{\hat{c}(t)}{\hat{k}(t)} - \delta - \left(-\frac{1}{1+\eta}\frac{\dot{\hat{c}}(t)}{\hat{c}(t)} + \frac{\alpha}{1+\eta}\frac{\dot{\hat{k}}(t)}{\hat{k}(t)}\right),
\end{aligned}$$

then rearrange to get

$$\begin{aligned}
\frac{\dot{\hat{c}}(t)}{\hat{c}(t)} &= \alpha\left[\hat{k}(t)\right]^{\alpha-1} + \frac{\alpha}{\alpha+\eta}\frac{\hat{c}(t)}{\hat{k}(t)} + \frac{\alpha\delta}{\alpha+\eta} - \frac{(\rho+\delta)(1+\alpha+\eta)}{\alpha+\eta}, \\
\frac{\dot{\hat{k}}(t)}{\hat{k}(t)} &= \hat{k}(t)^{\alpha-1} - \frac{\eta}{\alpha+\eta}\frac{\hat{c}(t)}{\hat{k}(t)} - \frac{\eta\delta}{\alpha+\eta} - \frac{\rho+\delta}{\alpha+\eta},
\end{aligned}$$

Rewrite these two equations into log-linearized form

$$\begin{aligned}
\frac{d\ln\hat{c}}{dt} &= \alpha\exp\left[(\alpha-1)\ln\hat{k}\right] + \frac{\alpha}{\alpha+\eta}\exp\left[\ln\left(\frac{\hat{c}}{\hat{k}}\right)\right] \\
&\quad + \frac{\alpha\delta}{\alpha+\eta} - \frac{(\rho+\delta)(1+\alpha+\eta)}{\alpha+\eta}, \\
\frac{d\ln\hat{k}}{dt} &= \exp\left[(\alpha-1)\ln\hat{k}\right] - \frac{\eta}{\alpha+\eta}\exp\left[\ln\left(\frac{\hat{c}}{\hat{k}}\right)\right] - \frac{\eta\delta}{\alpha+\eta} - \frac{\rho+\delta}{\alpha+\eta}.
\end{aligned}$$

Applying first order Taylor expansion to these two equations around steady state, it's simple to get $\frac{\dot{\hat{c}}}{\hat{c}}$

$$
\begin{aligned}
\frac{d\ln\hat{c}}{dt} &= \left\{\alpha(\alpha-1)\exp\left[(\alpha-1)\ln\hat{k}^*\right]-\frac{\alpha}{\alpha+\eta}\exp\left[\ln\left(\frac{\hat{c}^*}{\hat{k}^*}\right)\right]\right\}\left[\ln\left(\frac{\hat{k}}{\hat{k}^*}\right)\right] \\
&\quad +\frac{\alpha}{\alpha+\eta}\exp\left[\ln\left(\frac{\hat{c}^*}{\hat{k}^*}\right)\right]\left[\ln\left(\frac{\hat{c}}{\hat{c}^*}\right)\right] \\
&= \left[(\alpha-1)(\rho+\delta)-\frac{\alpha}{\alpha+\eta}\left(\frac{\rho+\delta}{\alpha}-\delta\right)\right]\left[\ln\left(\frac{\hat{k}}{\hat{k}^*}\right)\right] \\
&\quad +\frac{\alpha}{\alpha+\eta}\left(\frac{\rho+\delta}{\alpha}-\delta\right)\left[\ln\left(\frac{\hat{c}}{\hat{c}^*}\right)\right],
\end{aligned}
$$

as well as $\frac{\dot{\hat{k}}}{\hat{k}}$

$$
\begin{aligned}
\frac{d\ln\hat{k}}{dt} &= \left\{(\alpha-1)\exp\left[(\alpha-1)\ln\hat{k}^*\right]+\frac{\eta}{\alpha+\eta}\exp\left[\ln\left(\frac{\hat{c}^*}{\hat{k}^*}\right)\right]\right\}\left[\ln\left(\frac{\hat{k}}{\hat{k}^*}\right)\right] \\
&\quad -\frac{\eta}{\alpha+\eta}\exp\left[\ln\left(\frac{\hat{c}^*}{\hat{k}^*}\right)\right]\left[\ln\left(\frac{\hat{c}}{\hat{c}^*}\right)\right] \\
&= \frac{(\alpha-1)(\rho+\delta)+\rho\eta}{\alpha+\eta}\left[\ln\left(\frac{\hat{k}}{\hat{k}^*}\right)\right]-\frac{\eta}{\alpha+\eta}\left(\frac{\rho+\delta}{\alpha}-\delta\right)\left[\ln\left(\frac{\hat{c}}{\hat{c}^*}\right)\right].
\end{aligned}
$$

For simplicity, rewrite the linearized system as

$$
\begin{aligned}
\begin{bmatrix} \frac{d\ln\hat{k}}{dt} \\ \frac{d\ln\hat{c}}{dt} \end{bmatrix} &= \begin{bmatrix} \frac{(\alpha-1)(\rho+\delta)+\rho\eta}{\alpha+\eta} & -\frac{\eta}{\alpha+\eta}\left(\frac{\rho+\delta}{\alpha}-\delta\right) \\ \left[(\alpha-1)(\rho+\delta)-\frac{\alpha}{\alpha+\eta}\left(\frac{\rho+\delta}{\alpha}-\delta\right)\right] & \frac{\alpha}{\alpha+\eta}\left(\frac{\rho+\delta}{\alpha}-\delta\right) \end{bmatrix}\begin{bmatrix} \ln\left(\frac{\hat{k}}{\hat{k}^*}\right) \\ \ln\left(\frac{\hat{c}}{\hat{c}^*}\right) \end{bmatrix} \\
&= \mathbf{A}\begin{bmatrix} \ln\left(\frac{\hat{k}}{\hat{k}^*}\right) \\ \ln\left(\frac{\hat{c}}{\hat{c}^*}\right) \end{bmatrix}.
\end{aligned}
$$

To find the eigenvalues of matrix $\mathbf{A}$, solve

$$
\begin{vmatrix} \frac{(\alpha-1)(\rho+\delta)+\rho\eta}{\alpha+\eta}-\lambda & -\frac{\eta}{\alpha+\eta}\left(\frac{\rho+\delta}{\alpha}-\delta\right) \\ \left[(\alpha-1)(\rho+\delta)-\frac{\alpha}{\alpha+\eta}\left(\frac{\rho+\delta}{\alpha}-\delta\right)\right] & \frac{\alpha}{\alpha+\eta}\left(\frac{\rho+\delta}{\alpha}-\delta\right)-\lambda \end{vmatrix} = 0
$$

for λ and this gives

$$
\lambda_1 = \frac{\rho-\sqrt{\rho^2+4\left(\frac{1+\eta}{\alpha+\eta}\right)(1-\alpha)(\rho+\delta)\left(\frac{\rho+\delta}{\alpha}-\delta\right)}}{2} < 0, \tag{5}
$$

$$\lambda_2 = \frac{\rho + \sqrt{\rho^2 + 4\left(\frac{1+\eta}{\alpha+\eta}\right)(1-\alpha)(\rho+\delta)\left(\frac{\rho+\delta}{\alpha} - \delta\right)}}{2} > 0, \tag{6}$$

showing the existence of saddle path (λ_1 is the stable solution). Moreover notice that λ_1 differs from λ_{1R} only by the term $\frac{1+\eta}{\alpha+\eta}$, and this term is larger than 1 since $\alpha < 1$. Then

$$|\lambda_1| > |\lambda_{1R}|,$$

suggesting that the speeds of convergence for $\hat{c}(t)$ and $\hat{k}(t)$ are higher than those for the standard Ramsey model.