8

Evil is the Root of All Money

"What this country needs,"he (Mr. President Hoover) told Christopher Morley, "is a great poem."To Rudy Vallee he said in the spring of 1932, "If you can sing a song that would make people forget the Depression, I'll give you a medal."Vallee didn't get the medal. Instead he sang:

They used to tell me I was building a dream
And so I followed the mob.
When there was earth to plough or guns to bear
I was always there right on the job.

Once I built a railroad, made it run
Made it race against time.
Once I built a railroad, now it's done.
Brother, can you spare a dime?

—William Manchester (1973), *The Glory and the Dream*

1 Introduction

Finally we come to the second pillar of this course — money. As you may have already heard thousands of times from many macro textbooks, money is just anything generally acceptable as a mean of exchange, valued, at least in part, because of its role in the exchange process. You may have further heard of different types of monies — *Inside money* is a liability of the private economy, i.e., of private institutions. Between the liabilities there are demand deposits, i.e., checking accounts, and saving accounts. Their value is backed by the value of assets: loans, securities and reserves, which are a fraction of liabilities. It is considered as the "output"of the banking sector and of the financial sector. *Outside money* is a liability of the government. Its key feature is that it is unbacked. It has value only as a mean of exchange and it has only an exchange value. Its value is the amount of goods and purchases one can get in exchange — the inverse of the price level. The purchasing power of outside money is $\frac{1}{P_t}$, in which P_t is the price level at time t. $\frac{P_{t+1}-P_t}{P_t}$ is the rate of inflation — although such dichotomy of money is extremely important in linking macro & finance and its surrounding literature is booming , in this course we will mostly regard money as outside, i.e. some intrinsically worthless paper printed by the government without being backed by the value of assets.

But wait. The models we learned before today, mainly the neoclassical growth models, are models of non-monetary economy. There is no transactional role for money in these models, and what makes it worse is that even we introduce money (essentially paper) in these models, it simply has a zero nominal rate of return and is therefore dominated in rate of return by other interest bearing assets — i.e. nobody would keep it at all! Therefore the key that bridges what we have learnd and monetary economics is to specify the role for money so that people are willing to hold positive quantities of money, i.e. money must be justified to own a positive value. And only after this is justified can we say anything about monetary policy.

Section 2, trying to award money its value by adding it into the utility functions following Sidrauski (1967), is essentially a reader for Walsh (2010) Chapter 2. The chapter by Walsh is an excellent, full-fledged must-read of money-in-the-utility model, however, readers may feel hard to follow. One reason is the specification of the optimization problem is a hidden myth there, so people may wonder how one can suddenly arrive at those fabulous long equations. The other reason is that the solution procedure is done by dynamic programming method — although this is definitely the main-stream (and here more than enough) approach, readers may easily get stuck in the quagmire of the first order as well as envelope conditions. So in the beginning of Section 2 I take the setups from Walsh, but then quickly deviate from the approach of him by uncovering the hidden tricks in better characterizing the problem and taking the solution menu of optimal control. Hope this could make the readers a more pleasant voyage through the jungle of Walsh (2010).

Section 3 briefly presents another model by imposing an additional money balance constraint, leading to very similar results as those in the former one. However, we are not able to cover more due to the limit of time, and readers will find some interesting readings listed in the end of this chapter.

2 Money in the Utility

Money in the utility models assume that money yields direct utility by incorporating money balances directly into the utility functions of the representative agent (think why such seemingly naïve specification is in fact plausible and useful), i.e. in comparison to the models we learned before, a representative agent gains her utility from both real consumption and real money balance,

$$U_t = u(c_t, m_t)$$

in which m_t is the real money balance equal to the nominal amount of money divided by the price level in each period t. To make money offer some non-pecuniary benefit, assume that $u(\cdot)$ is strictly concave in both c_t and m_t. And in order to make it a general equilibrium model, we introduce bonds bearing a nominal interest rate as well as physical capital for the production in the economy. Then in a dynamic context, the agent's problem is to

$$\max_{\{k_t, c_t, b_t, m_t\}_{t=0}^{+\infty}} U = \sum_{t=0}^{+\infty} \beta^t u(c_t, m_t)$$

in which b_t is the real bonds balance equal to the nominal amount of bonds divided by the price level in each period t. And as a convention, people use the small letters for per capita variables.

2.1 *Specification of the Optimization Problem*

Similar as what we did in the Ramsey problem, now we define the law of motion to complete the specification of the optimization problem. To make it simpler we can start from defining the social resource constraint of a benevolent central planner [1]

$$Y_t + \tau_t N_t + (1-\delta)K_{t-1} + \frac{(1+i_{t-1})B_{t-1}}{P_t} + \frac{M_{t-1}}{P_t} = C_t + K_t + \frac{M_t}{P_t} + \frac{B_t}{P_t} \tag{1}$$

in which Y_t is the aggregate output, K_{t-1} is the aggregate stock of capital at the beginning of period t, and $\tau_t N_t$ is the aggregate real value of the lump-sum tax income (per capita tax times population).

The timing of the model is a little bit tricky and plays an important role in the specification of (1) which drives the final results. Though seemingly complicated, the intuition behind

[1] As you learned in the Ramsey-Cass-Koopmans model, this is equivalent to the approach of finding the resource constraint of a decentralized economy. You may try to start from there and see the different reasoning — as in Chapter 2.2, Walsh (2010).

equation (1) is pretty clear: In each period the task of the central planner is divided into two steps

- She collets all the resources available in the beginning of the period as the left hand side of (1) shows — Implement the production using the *past period* capital (note it is slightly different from the standard Ramsey-Cass-Koopmans model, and you will see this makes life easier in finding the solution) via neoclassical technology (assuming full employment, i.e. the labor input L_t is equal to the population N_t. To make it simple, assume that the population grows at a constant rate n, i.e. $N_t = N_0(1+n)^t$)

  $$Y_t = F(K_{t-1}, N_t),$$

  tax the households, collect the remaining capital after depreciation, and determine the *real* balance (because the production, tax and capital are expressed in *real* terms!) of bonds as well as money in the economy.
- Then she decides how to allocate these resources as the right hand side of (1) shows — distribute the consumption goods to the households, prepare the capital input for the next period, and adjust the real balance of bonds as well as money for the current period.

Now the same trick as before — find the law of motion in per capita terms. Manipulate (1)

$$\begin{aligned}
&F\left(K_{t-1}, N_{t-1}(1+n)\right) + \tau_t N_t + (1-\delta)k_{t-1}\frac{N_t}{1+n} + \frac{(1+i_{t-1})B_{t-1}N_t + M_{t-1}N_t}{N_{t-1}P_{t-1}(1+n)(1+\pi_t)} \\
=&N_t F\left(\frac{K_{t-1}}{N_{t-1}(1+n)}, 1\right) + \tau_t N_t + (1-\delta)k_{t-1}\frac{N_t}{1+n} + \frac{\left[(1+i_{t-1})B_{t-1} + M_{t-1}\right]N_t}{N_{t-1}P_{t-1}(1+n)(1+\pi_t)} \\
=&N_t c_t + N_t k_t + \frac{M_t N_t}{N_t P_t} + \frac{B_t N_t}{N_t P_t},
\end{aligned}$$

and define $b_t = \frac{B_t}{N_t P_t}$ as per capita bond balance as well as $m_t = \frac{M_t}{N_t P_t}$ as per capita money balance, by dividing both sides with N_t the equation above can be rewritten as

$$f\left(\frac{k_{t-1}}{1+n}\right) + \tau_t + \frac{(1-\delta)k_{t-1}}{1+n} + \frac{(1+i_{t-1})b_{t-1} + m_{t-1}}{(1+n)(1+\pi_t)} = c_t + k_t + m_t + b_t. \tag{2}$$

Then the representative agent's problem can be expressed as

$$\begin{aligned}
&\max_{\{k_t, c_t, b_t, m_t\}_{t=0}^{+\infty}} U = \sum_{t=0}^{+\infty} \beta^t u(c_t, m_t), \\
&\quad s.t.\ f\left(\frac{k_{t-1}}{1+n}\right) + \tau_t + \frac{(1-\delta)k_{t-1}}{1+n} + \frac{(1+i_{t-1})b_{t-1} + m_{t-1}}{(1+n)(1+\pi_t)} = c_t + k_t + m_t + b_t.
\end{aligned}$$

2.2 Getting the First Order Conditions

What's next? No matter you want to go on with optimal control method (as I will do) or dynamic programming (as Carl Walsh does), you have to distinguish the state variables from the control variables. But this seems not that clear at the first glance here, because it's difficult to say which variable(s) fully characterize(s) the state the decision maker faces in the beginning of each period and hence hard to find the transition function(s). Now you can understand why we set up the timing of the events in such a way which is described as a two-step setup in the beginning of this section: By assuming that the social planner collects all the available resources in the beginning of each period and then allocates what she gets, we actually already defined a state variable — the left hand side of (2)! This is exactly what the allocation decision is based on! We define it as the *per capita wealth*, w_t, in each period t, such as

$$w_t = f\left(\frac{k_{t-1}}{1+n}\right) + \tau_t + \frac{(1-\delta)k_{t-1}}{1+n} + \frac{(1+i_{t-1})b_{t-1} + m_{t-1}}{(1+n)(1+\pi_t)} = c_t + k_t + m_t + b_t. \tag{3}$$

Then what's the control variable? From the representative agent's problem we see that she has to find the optimal path for $\{k_t, c_t, b_t, m_t\}_{t=0}^{+\infty}$. But the sum of these four variables is just w_t. Therefore we can define any three out of four as control variables — say, c_t ,b_t and m_t. Now rewrite the law of motion, or the transition function, in terms of the state and control variables by manipulating (3)

$$\begin{aligned} w_{t+1} - w_t &= f\left(\frac{k_t}{1+n}\right) + \tau_{t+1} + \frac{(1-\delta)k_t}{1+n} + \frac{(1+i_t)b_t + m_t}{(1+n)(1+\pi_{t+1})} - w_t \\ &= f\left(\frac{w_t - c_t - m_t - b_t}{1+n}\right) + \tau_{t+1} + \frac{(1-\delta)(w_t - c_t - m_t - b_t)}{1+n} \\ &\quad + \frac{(1+i_t)b_t + m_t}{(1+n)(1+\pi_{t+1})} - w_t, \end{aligned}$$

with which we can set up the present value Hamiltonian

$$\begin{aligned} \mathcal{H}_t =& \beta^t u(c_t, m_t) + \lambda_t \left[f\left(\frac{w_t - c_t - m_t - b_t}{1+n}\right) + \tau_{t+1} + \frac{(1-\delta)(w_t - c_t - m_t - b_t)}{1+n} \right. \\ & \left. + \frac{(1+i_t)b_t + m_t}{(1+n)(1+\pi_{t+1})} - w_t \right]. \end{aligned}$$

The first order conditions are

$$\frac{\partial \mathcal{H}_t}{\partial c_t} = \beta^t \frac{\partial u(c_t, m_t)}{\partial c_t} + \lambda_t \left[-\frac{1}{1+n} f'\left(\frac{w_t - c_t - m_t - b_t}{1+n}\right) - \frac{1-\delta}{1+n} \right] = 0, \tag{4}$$

$$\frac{\partial \mathcal{H}_t}{\partial m_t} = \beta^t \frac{\partial u(c_t, m_t)}{\partial m_t} + \lambda_t \left[-\frac{1}{1+n} f' \left(\frac{w_t - c_t - m_t - b_t}{1+n} \right) - \frac{1-\delta}{1+n} + \frac{1}{(1+n)(1+\pi_{t+1})} \right] = 0, \tag{5}$$

$$\frac{\partial \mathcal{H}_t}{\partial b_t} = \lambda_t \left[-\frac{1}{1+n} f' \left(\frac{w_t - c_t - m_t - b_t}{1+n} \right) - \frac{1-\delta}{1+n} + \frac{1+i_t}{(1+n)(1+\pi_{t+1})} \right] = 0, \tag{6}$$

$$\frac{\partial \mathcal{H}_t}{\partial w_t} = \lambda_t \left[\frac{1}{1+n} f' \left(\frac{w_t - c_t - m_t - b_t}{1+n} \right) + \frac{1-\delta}{1+n} - 1 \right] = -(\lambda_t - \lambda_{t-1}), \tag{7}$$

as well as the transversality condition

$$\lim_{T \to +\infty} \lambda_T w_T = 0. \tag{8}$$

By equation (6) one can see that

$$-\frac{1}{1+n} f' \left(\frac{w_t - c_t - m_t - b_t}{1+n} \right) - \frac{1-\delta}{1+n} + \frac{1+i_t}{(1+n)(1+\pi_{t+1})} = 0. \tag{9}$$

Use this fact to rewrite the first order conditions

$$\frac{\partial \mathcal{H}_t}{\partial c_t} = \beta^t \frac{\partial u(c_t, m_t)}{\partial c_t} + \lambda_t \left[-\frac{1+i_t}{(1+n)(1+\pi_{t+1})} \right] = 0, \tag{10}$$

$$\frac{\partial \mathcal{H}_t}{\partial m_t} = \beta^t \frac{\partial u(c_t, m_t)}{\partial m_t} + \lambda_t \left[-\frac{i_t}{(1+n)(1+\pi_{t+1})} \right] = 0, \tag{11}$$

$$\frac{\partial \mathcal{H}_t}{\partial w_t} = \lambda_t \left[\frac{1+i_t}{(1+n)(1+\pi_{t+1})} - 1 \right] = -(\lambda_t - \lambda_{t-1}). \tag{12}$$

Now we are approaching the final results with the reshaped first order conditions (9) – (12) . First, continue with (9)

$$\begin{aligned} \frac{1+i_t}{(1+n)(1+\pi_{t+1})} &= \frac{1}{1+n} \left[f' \left(\frac{w_t - c_t - m_t - b_t}{1+n} \right) + 1 - \delta \right], \\ \frac{1+i_t}{1+\pi_{t+1}} &= 1 + f'_t - \delta \end{aligned}$$

in which f'_t denotes the productivity at period t. Then quickly we realize that $f'_t - \delta$ is just the real interest rate for period t, r_t, in general equilibrium. Therefore

$$1 + i_t = (1 + r_t)(1 + \pi_{t+1}). \tag{13}$$

Take logarithm on both sides and notice that the rates are small numbers, we get

$$i_t = r_t + \pi_{t+1}. \tag{14}$$

Second, divide (11) by (10) one can see that

$$\frac{\frac{\partial u(c_t,m_t)}{\partial m_t}}{\frac{\partial u(c_t,m_t)}{\partial c_t}} = \frac{i_t}{1+i_t}. \tag{15}$$

Third, notice that equation (12) can be simplified as

$$\lambda_t \left[\frac{1+i_t}{(1+n)(1+\pi_{t+1})} \right] = \lambda_{t-1}, \tag{16}$$

and by Equation (10) the left hand side of it is just

$$\lambda_t \left[\frac{1+i_t}{(1+n)(1+\pi_{t+1})} \right] = \beta^t \frac{\partial u(c_t, m_t)}{\partial c_t}. \tag{17}$$

Combine these two equations (16) and (17) we get

$$\beta^t \frac{\partial u(c_t, m_t)}{\partial c_t} = \lambda_{t-1},$$

which is equivalent to

$$\beta^{t+1} \frac{\partial u(c_{t+1}, m_{t+1})}{\partial c_{t+1}} = \lambda_t \tag{18}$$

by one period update. Then we rewrite equation (10) as

$$\begin{aligned}
\beta^t \frac{\partial u(c_t, m_t)}{\partial c_t} &= \lambda_t \left[\frac{1+i_t}{(1+n)(1+\pi_{t+1})} \right] \\
&= \lambda_t \frac{1}{(1+n)(1+\pi_{t+1})} + \lambda_t \frac{i_t}{(1+n)(1+\pi_{t+1})} \\
&= \lambda_t \frac{1}{(1+n)(1+\pi_{t+1})} + \beta^t \frac{\partial u(c_t, m_t)}{\partial m_t}
\end{aligned}$$

in which the last step is simply taken from equation (11). Using (18) to eliminate λ_t we get

$$\beta^t \frac{\partial u(c_t, m_t)}{\partial c_t} = \beta^{t+1} \frac{\partial u(c_{t+1}, m_{t+1})}{\partial c_{t+1}} \frac{1}{(1+n)(1+\pi_{t+1})} + \beta^t \frac{\partial u(c_t, m_t)}{\partial m_t},$$

and simplify to obtain

$$\frac{\partial u(c_t, m_t)}{\partial c_t} = \frac{\beta \frac{\partial u(c_{t+1}, m_{t+1})}{\partial c_{t+1}}}{(1+n)(1+\pi_{t+1})} + \frac{\partial u(c_t, m_t)}{\partial m_t}. \tag{19}$$

As a last step we can also directly insert (18) into (10)

$$\beta^t \frac{\partial u(c_t, m_t)}{\partial c_t} = \beta^{t+1} \frac{\partial u(c_{t+1}, m_{t+1})}{\partial c_{t+1}} \left[\frac{1+i_t}{(1+n)(1+\pi_{t+1})} \right],$$

and simplify to obtain

$$\frac{\frac{\partial u(c_t, m_t)}{\partial c_t}}{\frac{\partial u(c_{t+1}, m_{t+1})}{\partial c_{t+1}}} = \beta \frac{1+i_t}{(1+n)(1+\pi_{t+1})}. \tag{20}$$

Insert (13) into (20) and we get

$$\frac{\frac{\partial u(c_t, m_t)}{\partial c_t}}{\frac{\partial u(c_{t+1}, m_{t+1})}{\partial c_{t+1}}} = \beta \frac{1+r_t}{1+n}. \tag{21}$$

2.3 Interpreting the Results

We pick up the central results from the lengthy derivation above and list them as following:

$$1 + i_t = (1 + r_t)(1 + \pi_{t+1}), \tag{22}$$

$$\frac{\frac{\partial u(c_t, m_t)}{\partial m_t}}{\frac{\partial u(c_t, m_t)}{\partial c_t}} = \frac{i_t}{1+i_t}, \tag{23}$$

$$\frac{\partial u(c_t, m_t)}{\partial c_t} = \frac{\beta \frac{\partial u(c_{t+1}, m_{t+1})}{\partial c_{t+1}}}{(1+n)(1+\pi_{t+1})} + \frac{\partial u(c_t, m_t)}{\partial m_t}, \tag{24}$$

$$\frac{\frac{\partial u(c_t, m_t)}{\partial c_t}}{\frac{\partial u(c_{t+1}, m_{t+1})}{\partial c_{t+1}}} = \beta \frac{1+r_t}{1+n}. \tag{25}$$

These equations have straightforward interpretations. Since resources are divided between consumption, capital, bonds, and money balances, each of them must yield the same marginal benefit for the agent at the optima.

2.3.1 The Fisher Parity

Equation (22) links the nominal return on bonds, inflation, and the real return on capital, called the *Fisher parity* after Irving Fisher. It shows the gross nominal rate of interest equals to the gross real return on capital times one plus the expected rate of inflation.

2.3.2 Money Demand Function

Equation (23) is the money demand function, showing that the marginal utility of real money balances normalized by the marginal utility from consumption (or, to put it technically, the marginal rate of substitution between money and consumption) is equal to the relative price of real money balances in terms of the consumption goods, i.e. the opportunity cost of holding money.

To make it clearer, we further explore equation (23) by assuming that the utility function $u(c_t, m_t)$ is additively separable, i.e.

$$\frac{\partial^2 u(c_t, m_t)}{\partial c_t \partial m_t} = 0,$$

to separate the effects of real money balance and consumption on the utility function (and we keep this assumption for the rest of this section). Now consider two situations:

- Suppose that there is an increase of i_t, then the right hand side of equation (23) goes up and $\frac{\partial u(c_t,m_t)}{\partial m_t}$ has to go up (holding c_t constant). Given that the utility function is strictly concave in m_t, the real money balance m_t must go down — because of the opportunity cost of holding money goes up (remember that money pays no interest rate). Therefore there is an inverse relationship between m_t and i_t;
- On the other hand suppose that i_t is kept constant and there is an increase in c_t, then $\frac{\partial u(c_t,m_t)}{\partial c_t}$ goes down. To maintain the equality $\frac{\partial u(c_t,m_t)}{\partial m_t}$ has to go down as well, meaning that m_t goes up — the real money balance moves in the same direction as consumption c_t.

In summary the real money demand depends on the level of consumption as well as the nominal interest rate, which can be captured in the following equation that is often seen in the intermediate macro textbooks

$$\frac{M_t}{P_t} = L(i_t, c_t) \text{ with } \frac{\partial L}{\partial i_t} < 0 \text{ and } \frac{\partial L}{\partial c_t} > 0.$$

And this is the basis of the well-known *LM* curve [2] .

2.3.3 Marginal Benefit of Money Holding

As argued in the beginning of SECTION 2.3, *in the margin each additional unit of consumption, capital, bonds, and money holdings must yield the same marginal benefit for the agent at the*

[2] See further discussion in Romer (2006), CHAPTER 5.1 THE MONEY MARKET.

optima, otherwise it cannot be an equilibrium solution. Equation (24) exactly captures such balance between money holding and consumption. In the margin adding one unit money holding has two kinds of benefits:

- The additional money directly generates some additional utility today, i.e. $\frac{\partial u(c_t,m_t)}{\partial m_t}$;
- Indirectly, the additional money today will be included in the initial wealth of tomorrow and generates some additional utility in the future. How can we compute such future benefit? Again, *in the margin each additional unit of consumption, capital, bonds, and money holdings must yield the same marginal benefit for the agent at the optima* and this holds for *each period*. So we can easily compute the future benefit by simply allocate this additional unit of period $t+1$ initial wealth for c_{t+1}. In the computation what we have to take care of are
  - One additional unit of per capita real money holding today is not equal to an additional unit of per capita initial wealth tomorrow, instead, it should be discounted by the inflation $1+\pi_{t+1}$ and diluted by the population growth $1+n$;
  - The marginal utility tomorrow should be discounted by the factor β.

The sum of these two kinds of benefits is expressed in the right hand side of equation (24), which should be equal to the marginal utility gained from consumption — if instead this additional unit of money holding is consumed today.

2.3.4 Real Interest Rate Determination

Equation (25) pins down the real interest rate in optima. Rewrite this equation as

$$\begin{aligned} 1+r_t &= (1+n)\frac{1}{\beta}\frac{\frac{\partial u(c_t,m_t)}{\partial c_t}}{\frac{\partial u(c_{t+1},m_{t+1})}{\partial c_{t+1}}} \\ &= (1+n)(1+\rho)\frac{\frac{\partial u(c_t,m_t)}{\partial c_t}}{\frac{\partial u(c_{t+1},m_{t+1})}{\partial c_{t+1}}}, \end{aligned}$$

showing that along the optimal path the gross real interest rate in each period must offset the pressures from the population growth, the discounting in the agent's preference and the growth in marginal utility with respect to the consumption.

2.4 Stationary Equilibrium Analysis

Now as a dynamic system, let's see what is going on in the steady state, in which both consumption and real money balances are constant, c^* and m^* (to make it simpler we assume that $n = 0$ from now on). Furthermore the bonds market equilibrium must be that

$$b_t = b^* = 0 \tag{26}$$

— everybody should be indifferent between borrowing and lending [3] . Then equations (22) – (25) can be rewritten as

$$1 + i_t = (1 + r_t)(1 + \pi_{t+1}), \tag{27}$$

$$\frac{\frac{\partial u(c^*,m^*)}{\partial m^*}}{\frac{\partial u(c^*,m^*)}{\partial c^*}} = \frac{i_t}{1 + i_t}, \tag{28}$$

$$1 = \frac{1 + r_t}{1 + \rho}, \tag{29}$$

plus the reshaped law of motion

$$f(k^*) + \tau^* + (1 - \delta)k^* + \frac{m^*}{1 + \pi^*} = c^* + k^* + m^*, \tag{30}$$

these four equations give a full characterization of the steady state. What's more, equation (29) implies that in the steady state r_t is constant and equal to ρ; then (28) implies that i_t is also constant; so as π_{t+1} from (27). From now on we can simply write i^*, r^* and π^*, dropping off the time notations. Now let's see what these equations mean.

2.4.1 The Neutrality of Money

That the real money balance m^* is constant in the steady state implies that $\frac{M_t}{P_t}$ should be constant. Also that the inflation rate π^* is constant means that $\frac{P_{t+1}}{P_t} = 1 + \pi^*$ is constant. From these two facts, we see that in the steady state

$$1 + \pi^* = \frac{P_{t+1}}{P_t} = \frac{M_{t+1}}{M_t} = 1 + \mu,$$

in which μ is the growth rate of money. Then equation (29) can be rewritten as

$$1 + \rho = \frac{1 + i^*}{1 + \mu}, \tag{31}$$

[3] Please note that the definition of a dynamic system's steady state only says that all the time derivatives are equal to 0 — in this case, it only means that the law of motion is time invariant, i.e. $w_t = w^* = c_t + k_t + m_t + b_t$, and $\frac{\partial u(c_t,m_t)}{\partial c_t}$ = constant (which implies that $c_t = c^*$ because the utility function is additively separable) from (25), which is NOT equivalent to that each of the other variables, k_t, m_t and b_t, is constant!
However, it's not difficult to show that w_t, c_t, k_t, m_t and b_t are in fact all constant in the steady state. From (25) one can see that r_t is constant in the steady state, implying that k_t is also constant (note that r_t is simply the marginal productivity of capital less the depreciation rate). Since $w_t = w^* = c_t+k_t+m_t+b_t$ and $b_t = b^* = 0$ in the steady state, it must be that $m_t = m^*$.

$$1 + i^* = \frac{1+\mu}{\beta}. \tag{32}$$

Equation (32) says that the steady state money growth rate and the agent's discount rate determine the nominal interest rate. This, further, through (28) pins down m^*.

What we see from these arguments is, the steady state is featured by the constant *real* money balance as well as the constant other *real* variables. This means that the changes in the *nominal* quantity of money are matched by proportional changes in the price levels, leaving the *real* money balance m^* constant and the other real variables unaffected. Such feature is called the *neutrality of money*.

We can go back into the equations to see why. Equation (28) is the key — Suppose that the nominal quantity of money M_t increases and P_t doesn't respond, then m_t increases, making the marginal utility of holding money $\frac{\partial u(c_t,m_t)}{\partial m_t}$ decreases. Therefore the agent would simply consume more to lower $\frac{\partial u(c_t,m_t)}{\partial c_t}$ — the pressure from the consumption finally pushes P_t to go up.

2.4.2 The Superneutrality of Money

We can find some more interesting feature if we continue to explore. Note that the only channel for the central planner to redistribute the tax income is money printing, i.e. the government's tax revenue is transferred to the representative agent via increasing her real money balance. Therefore in the steady state

$$\tau^* = m^* - \frac{m^*}{1+\pi^*} = \frac{\pi^* m^*}{1+\pi^*}.$$

Insert this into the law of motion (30), we see that

$$f(k^*) - \delta k^* = c^*. \tag{33}$$

Also we know that as a result of the optimal growth path

$$f'(k^*) - \delta = r^*. \tag{34}$$

Then equations (26), (28), (33) and (34) determine real values of all the variables in the steady state. A closer look suggests that the steady state here also exhibits a feature of the *superneutrality of money* — none of these variables is dependent on the rate of inflation π^*, or the growth rate of nominal money!

2.5 Extensions

The results and features of this model inspired infinite extensions and ignited numerous controversies. More discussions in Chapter 2.2 and 2.3, Walsh (2010). To be discussed in the class:

- The existence of equilibrium;
- Welfare maximization and optimal monetary policy;
- Extensions of the model.

3 Cash in Advance

Although it sounds pretty reasonable to include money in the utility function that money works via some mysterious mechanism which increases one's well-being, often people criticize such models in which money matters simply because it is assumed to be. Therefore people would like to show the value of money in situations where money is used. One way of doing this is to see how money facilitates transactions, assuming the transaction technology in the economy depends on money as a transaction medium, assuming that a certain amount of money has to be held in order to purchase the consumption goods. This type of models are generally called *cash-in-advance* models.

3.1 Specification of the Optimization Problem

The representative agent's objective is to choose a path for consumption and asset holdings to maximize

$$\sum_{t=0}^{+\infty} \beta^t u(c_t),$$

in which the agent's utility only comes from her consumption goods.

The timing of the model is featured by the following three-step stucture for each period t:

- In the beginning of the period the agent holds a certain amount of nominal money balance M_{t-1}, nominal bonds balance B_{t-1} (with an additional interest payment at the interest rate i_{t-1}) and capital stock k_{t-1} from the last period and receives a lump-sum nominal transfer TR_t from the government. Then the goods market opens, and the agent is only able to buy the goods for consumption with her holdings of money and transfer at the period's price level P_t, i.e. the agent's *cash-in-advance constraint* is

$$
\begin{aligned}
P_t c_t &\le M_{t-1} + TR_t, \\
c_t &\le \frac{M_{t-1}}{P_{t-1}} \frac{P_{t-1}}{P_t} + \frac{TR_t}{P_t} \\
&= \frac{m_{t-1}}{1+\pi_t} + \tau_t,
\end{aligned}
$$

in which the second step is just to express everything in real terms, and π_t is the inflation rate of period t. The agent consumes;

- Next the production is implemented via the neoclassical technology with k_{t-1} as input, and the capital depreciates at the rate δ;
- In the end the agent has to decide how much money, bonds and capital are to be allocated as the initial wealth of the next period.

One may find the settings here depart from those in the lectures in two ways:

- To make it simple, we don't distinguish between cash and credit goods here. The introduction of cash goods is useful to drive the wedge between the marginal utilities of the two types of goods, making consumers respond to the nominal interest rate as an opportunity cost of holding cash goods and generating the substitution between these two types of goods. However, our simplified version here is sufficient to obtain the results we are interested in;
- Here in each period the goods market opens first such that the consumers purchase consumption goods via the money holdings accummulated from the last period, in contrast to the model in the lectures where capital market opens first such that the consumers purchase consumption goods via the money holdings left from the capital transaction concerning K_t. However, in an economy without uncertainty, these two types of timing lead to the same results.

The overall budget constraint for the period can be thus expressed as

$$
\begin{aligned}
P_t \left[f(k_{t-1}) + (1-\delta)k_{t-1} \right] + M_{t-1} + TR_t + (1+i_{t-1})B_{t-1} &\ge P_t c_t + P_t k_t + M_t + B_t, \\
f(k_{t-1}) + (1-\delta)k_{t-1} + \frac{m_{t-1}}{1+\pi_t} + \tau_t + \frac{1+i_{t-1}}{1+\pi_t} b_{t-1} &\ge c_t + k_t + m_t + b_t.
\end{aligned}
$$

Again we express everything in real terms. The left hand side of the inequality is what she gains in the period, and the right hand side is what she has to spend.

Now the representative agent's problem can be characterized as

$$
\begin{aligned}
\max_{\{c_t, k_t, m_t, b_t\}_{t=0}^{+\infty}} \ & \sum_{t=0}^{+\infty} \beta^t u(c_t), \\
s.t. \ & f(k_{t-1}) + (1-\delta)k_{t-1} + \frac{m_{t-1}}{1+\pi_t} + \tau_t + \frac{1+i_{t-1}}{1+\pi_t} b_{t-1} \ge c_t + k_t + m_t + b_t, \\
& \frac{m_{t-1}}{1+\pi_t} + \tau_t \ge c_t.
\end{aligned}
$$

3.2 *Getting the First Order Conditions*

The optimization problem is a problem with several inequality constraints. Instead of worrying about the state / control variables, this time we simply start from the stone age approach – using Kuhn-Tucker Theorem.

Set up the Lagrangian for this problem

$$\mathscr{L} = \sum_{t=0}^{+\infty} \left\{ \beta^t u(c_t) + \lambda_t \left[f(k_{t-1}) + (1-\delta)k_{t-1} + \frac{m_{t-1}}{1+\pi_t} + \tau_t + \frac{1+i_{t-1}}{1+\pi_t} b_{t-1} - c_t - k_t - m_t - b_t \right] \right.$$
$$\left. + \mu_t \left[\frac{m_{t-1}}{1+\pi_t} + \tau_t - c_t \right] \right\},$$

in which λ_t and μ_t are the Lagrange multipliers for these two inequality constraints respectively.

The first order conditions $\forall t \in \{0, 1, 2, \ldots\}$ are

$$\frac{\partial \mathscr{L}}{\partial c_t} = \beta^t u'(c_t) - \lambda_t - \mu_t = 0, \tag{35}$$
$$\frac{\partial \mathscr{L}}{\partial k_t} = \lambda_{t+1} \left[f'(k_t) + (1-\delta) \right] - \lambda_t = 0, \tag{36}$$
$$\frac{\partial \mathscr{L}}{\partial m_t} = \frac{\lambda_{t+1}}{1+\pi_{t+1}} - \lambda_t + \frac{\mu_{t+1}}{1+\pi_{t+1}} = 0, \tag{37}$$
$$\frac{\partial \mathscr{L}}{\partial b_t} = \lambda_{t+1} \frac{1+i_t}{1+\pi_{t+1}} - \lambda_t = 0. \tag{38}$$

From (38) one can get the relationship between λs of the neighboring periods

$$\lambda_t = \frac{1+i_t}{1+\pi_{t+1}} \lambda_{t+1}. \tag{39}$$

From (36) one can also get the relationship between λs of the neighboring periods

$$\lambda_t = \lambda_{t+1} \left[f'(k_t) + (1-\delta) \right], \tag{40}$$
$$\lambda_t = \lambda_{t+1}(1+r_t) \tag{41}$$

using the fact that $r_t = f'(k_t) - \delta$. Then combining the results of (41)and (39)

$$1 + i_t = (1+r_t)(1+\pi_{t+1}), \tag{42}$$
$$i_t \approx r_t + \pi_{t+1}. \tag{43}$$

And equations (42) and (43) again exhibit the Fisher Parity.

From (37) one can get the relationship between the two Lagrange multipliers

$$\lambda_t = \frac{\lambda_{t+1} + \mu_{t+1}}{1 + \pi_{t+1}}, \tag{44}$$

and insert equation (39) one can see that

$$\frac{1 + i_t}{1 + \pi_{t+1}}\lambda_{t+1} = \frac{\lambda_{t+1} + \mu_{t+1}}{1 + \pi_{t+1}}, \tag{45}$$

$$\mu_{t+1} = i_t\lambda_{t+1} \tag{46}$$

showing the intratemporal relationship between the two Lagrange multipliers.

3.3 Stationary Equilibrium Analysis

From (35) the discounted marginal utility of each period t is

$$\beta^t u'(c_t) = \lambda_t + \mu_t, \tag{47}$$

$$\beta^t u'(c_t) = \frac{1 + i_{t-1}}{i_{t-1}}\mu_t. \tag{48}$$

Notice that μ_t is the *present value* of the shadow price of the cash-in-advance constraint, which is the *current value*, denoted by $\tilde{\mu}_t$, discounted by β^t. Therefore (48) can be rewritten as

$$\beta^t u'(c_t) = \frac{1 + i_{t-1}}{i_{t-1}}\beta^t\tilde{\mu}_t, \tag{49}$$

$$\frac{\tilde{\mu}_t}{u'(c_t)} = \frac{i_{t-1}}{1 + i_{t-1}} \tag{50}$$

which in the steady can be written as

$$\frac{\tilde{\mu}^*}{u'(c^*)} = \frac{i_t^*}{1 + i_t^*}. \tag{51}$$

Note that equation (51) doesn't only possess the same form as equation (28), but also has exactly the same interpretation. Remember that the shadow price in the Lagrangian just means by how much the object fuction responses to a unit slackness in the constraint. Therefore $\tilde{\mu}^*$ here implies by how much the instantaneous utility fuction $u(c_t)$ responses if one unit of real money balance is added to the period t's cash-in-advance constraint, i.e. the marginal utility with respect to real money balance in the steady state $\frac{du(c^*)}{dm^*}$. Thus equation (51) has the same meaning as equation (28), reflecting the relative price of money holding and consumption goods.

One can find more such similarities. Insert (46) into (47)

$$\beta^t u'(c_t) = (1 + i_{t-1})\lambda_t, \tag{52}$$
$$\beta^{t+1} u'(c_{t+1}) = (1 + i_t)\lambda_{t+1}. \tag{53}$$

Divide (53) by (52)

$$\beta \frac{u'(c_{t+1})}{u'(c_t)} = \frac{1 + i_t}{1 + i_{t-1}} \frac{\lambda_{t+1}}{\lambda_t}. \tag{54}$$

Insert (41) into (54), using the fact that in the steady state $c_t = c^*$ and $i_t = i^*$ one can arrive at

$$1 = \frac{1 + \rho}{1 + r^*}, \tag{55}$$

which corresponds to equation (25).

Remember that one important feature of the money-in-the-utility model is the (super-)neutrality of money. Is this feature still maintained in current cash-in-advance model?

To answer this question, again we start from determining the steady state values of the real variables. First notice that in the optimum the cash-in-advance constraint is binding,

$$c_t = \frac{m_{t-1}}{1 + \pi_t} + \tau_t, \tag{56}$$

and in the steady state the transfer is entirely implemented by the injection of money, i.e.

$$\tau_t = \frac{M_t - M_{t-1}}{P_t}. \tag{57}$$

Combining (56) and (57) one can easily see that $c_t = m_t$. Therefore the steady state value of real money balance is also constant, $m_t = m^*$. By the same argument as Section 2.4.1 the current model also maintains the neutrality of money (and the steady state inflation rate is equal to the growth rate of money, $\pi^* = \mu$).

Furthermore, equation (55) determines that the steady state real interest rate only depends on an exogenous variable. This implies that the steady state capital stock is

$$k^* = (f')^{-1}(r^* + \delta). \tag{58}$$

Also by the same argument as Section 2.4.2 the steady state consumption is given by

$$c^* = f(k^*) - \delta k^*. \tag{59}$$

Equations (58) and (59) simply show the superneutrality of money holds here, for neither k^* nor c^* is affected by the inflation rate.

4 Readings

Walsh (2010), Chapter 2 – 3, Galí (2008), Chapter 2.

5 Bibliographic Notes

The money-in-the-utility model is based on Sidrauski (1967). The cash-in-advanced model originated from Clower (1967) and was formalized in Grandmont and Younes (1972). The model presented here is a much simplified version of Lucas (1980) and Svensson (1985), the timing struction following the convention of the latter.

Both Chapter 4, Blanchard and Fischer (1989) and Chapter 3, Walsh (2010) include rich surveys of the other models exploring the roles of money, such as Baumol (1952) & Tobin (1956) type of shopping-time models and so on. The reason why the latter book is recommended in the readings is because the former is mostly based on the overlapping generation model which is gradually losing its popularity in current macro studies, for in such model money essentially functions as a medium for storage and people often find it hard to reconcile empirical works in this framework (however, some interesting researches, both theoretical and empirical, based on the Blanchard-Yaari type of overlapping generation models are still blossoming). Nevertheless, Blanchard and Fischer (1989) is still recommended as an excellent reading for the same issues explored in an alternative framework. Galí (2008), Chapter 2, provides a concise, pedagogical, yet state-of-art summary of baseline monetary models.

Current chapter is a pivotal leap from neoclassical growth theories towards the monetary economics. Since a large share of modern macro studies are actually based on the samilar settings, it's important to understand the features of these models in order to see how people manage to reach the results they desire by extending the prototype models in different ways. On the other hand, the presented models, explaining the roles of money without introducing short-run rigidities, simply bring out more troubles than the questions they solve, especially when people attempt to see the policy implications (for example, the famous Friedman rule, which is in fact a very robust phenomenon in these models). There is a huge literature trying to get around the problems in different dimensions, e.g. Chari, Christiano and Kehoe (1991, 1996), Schmitt-Grohé and Uribe (2004a, 2004b and 2005), etc. We will come across these issues later.

6 Exercises

6.1 Money in the Utility: Equilibrium and Stability

Consider an infinitely lived agent with utility function

$$\int_0^{+\infty} [c(t) + V(m(t))]\, e^{-\rho t} dt,$$

where c is consumption, m are real money holdings, and V is an increasing and concave function. Money is the only asset. Income is exogenously given by $y(t)$.

a)$^{\mathrm{A}}$ Formulate the transition equation in real balances (money holdings).

b)$^{\mathrm{A}}$ Formulate the Hamiltonian and first order conditions.

c)$^{\mathrm{A}}$ The growth rate of nominal money supply is given by μ. Derive a differential equation describing the optimal real balances.

d)$^{\mathrm{C}}$ Discuss potential steady state equilibria and their stability. Characterize conditions that rule out hyperinflationary bubbles.

e)$^{\mathrm{B}}$ Discuss the special case of $V(m) = m^{\alpha}$.

6.2 Money in the Utility

Consider a discrete version of Sidrauski's money in the utility approach: An infinitely lived representative agent maximizes discounted life time utility

$$\sum_{t=0}^{+\infty} \beta^t U(c_t, m_t)$$

with $\beta \in (0,1)$ as discount rate, c_t consumption and $m_t = \frac{M_t}{P_t}$ as real money balances. Each period, the agent is endowed with y_t. y_t can be used for private or government consumption: $y_t = c_t + g_t$. Initially, the agent owns the money stock M_0 and one period nominal government bonds B_0 . Period t bonds B_t yield a return i_t. The government finances g_t via taxes τ_t , seigniorage or government bonds.

a)[A] Formulate the period budget constraint of both the agent and the government and derive the present value budget constraint.

b)[A] Characterize the first-order conditions for the agent's optimal path.

c)[B] Show that with additive separable preferences $U(c_t, m_t) = u(c_t) + v(m_t)$, the real rate of interest depends only on the time path of the real resources available for consumption.

d)[C] Assume that $U(c_t, m_t) = c_t^\alpha + m_t^\alpha$. Derive the money demand function $m(c_t, i_t)$ and characterize elasticity with respect to c_t and i_t . Show why the price level may not be determinate if the central bank pegs the interest rate to a fixed level $i_t = \bar{i}$.

e)[C] Assume that both endowment and government spending are constant: $y_t = y$; $g_t = g$. Characterize conditions for steady state. Show that the Friedman rule maximizes per period utility. Discuss reasons why this rule may not be optimal in a more general setting.

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