4

Here and There

Sometimes one's city can look like an alient place. Streets that seem like home will suddenly change colour; I'll look into the ever mysterious crowds pressing past me and suddenly think they've been there for hundreds of years. With its muddy parks and desolate open spaces, its electricity poles and the billboards plastered over its squares and its concrete monstrosities, this city, like my soul, is fast becoming an empty – a *very* empty – place.

—Orhan Pamuk (2005), *Istanbul: Memories and the City*

1 Introduction

In this class we examine a couple of topics of growth modelling, each of which is not big enough to fill up a single chapter.

The first two sections discuss several issues concerning the Ramsey-Cass-Koopmans model, which readers may feel eager to know but are hardly able to find in the textbooks. One is how people should interpret the equivalence in the results obtained from two different approaches, i.e. to solve Ramsey-Cass-Koopmans model as a central planning problem or by finding competitive equilibrium allocations. We show that in special cases the two results do replicate each other, however, for most of the times one has to keep in mind whose decision the policy affects and it would be safe to "render to the Caesar the things that are Caesar's and to God the things that are God's"without mixing up these two approaches.

The other is analysing the economy's qualitative response to the policy change using phase diagram. Although the business of macro study has been dominated by the quantitative (numerical) methods, it definitely makes sense to think qualitatively before going with the numbers. As we can see, qualitative method provides faster (and often sufficiently precise) predictions, and people may use it to check whether they got everything correct in numerical simulations.

The last topic is a portrait of the overlapping generation model. It shows how a relatively generic model (with production and capital accumulation) is established and how people may proceed with it. However, from a pragmatic point of view, this section is more than enough for solving our exercises. Therefore, this part is more for your fun.

2 Ramsey-Cass-Koopmans Model: Centralized and Decentralized Versions

One can think about Ramsey-Cass-Koopmans problem in two different ways, (1) characterizing the maximizing behavior of an economy composed of distinct individual households, (2) finding the solution of a benevolent social planner who wants to maximize the social welfare. In this section, we will show that for standard Ramsey-Cass-Koopmans problem these two approaches result in the same solutions. However, such equivalence is more like a coincidence, and it's not always true that the solution from one approach replicates the solution from the other under the settings different from the standard problem.

2.1 The Central Planning Problem

Consider a central planning economy with neoclassical technology $Y(t) = F(K(t), L(t))$ (no technological progress), constant depreciation rate δ and constant population growth rate n. Then the standard textbook problem of a benevolent social planner is simply maximizing the utility of an infinitely lived representative agent,

$$\max_{c(t)} \int_0^{+\infty} e^{-\rho t} u(c(t)) dt, \tag{1}$$

given the flow budget constraint

$$\dot{k}(t) = f(k(t)) - (n + \delta)k(t) - c(t). \tag{2}$$

2.2 *Decentralized Ramsey-Cass-Koopmans Model*

Consider the problem of an individual infinitely living consumer who has some predetermined set of expectations for how the aggregate interest rate $r(t)$ and wage rate $w(t)$ will evolve. The consumer's object function is the same as (1). The household which she belongs to owns some assets $A(t)$, and can in principle also borrow (which can be expressed as a negative component of $A(t)$). The household's total net asset position at time t is therefore

$$\dot{A}(t) = w(t)L(t) + r(t)A(t) - C(t) \tag{3}$$

in which $L(t)$ is the amount of labor force (as well as population, because we assume that there's no unemployment) provided by the household, and $C(t)$ is the aggregate consumption level of the household. If we express the net asset position in per capita form with $a(t) = \frac{A(t)}{L(t)}$ and $c(t) = \frac{C(t)}{L(t)}$, hence $\frac{\dot{a}(t)}{a(t)} = \frac{\dot{A}(t)}{A(t)} - \frac{\dot{L}(t)}{L(t)}$ by log-linearization, then equation (3) can be transformed into

$$\dot{a}(t) = w(t) + r(t)a(t) - na(t) - c(t). \tag{4}$$

There are also competitive firms in this economy who want to maximize their profits. As explored in the first chapter of the class notes, the market prices for the labor and capital are

$$w(t) = f(k(t)) - k(t)f'(k(t)) \tag{5}$$

$$r(t) + \delta = f'(k(t)) \tag{6}$$

respectively. And furthermore the consumer is *representative* in the sense that she does not only represent the consumers but also represents the entrepreneurs, therefore

$$a(t) = k(t) \tag{7}$$

from both sides view. Insert (5), (6) and (7) into (4), one can see that

$$\dot{k}(t) = f(k(t)) - (n + \delta)k(t) - c(t), \tag{8}$$

which is exactly the same as (2)!

2.3 Beyond the Equivalence

Now one can see that both the social planner in a central planning economy and a representative consumer in a decentralized competitive economy face the same object function (1) and the same flow buget constraint, therefore there is no doubt that one can get the same solution following either approach.

Can we say anything more than such equivalence? Obviously from theoretical point of view this shows that market can work as well as benevolent social planners, so that we don't need people sitting in Moscow and planning for us. From practical point of view, it's much simpler to set up a social planner's problem than its decentralized version — Suppose that such equivalence could be justified, then one can solve a decentralized Ramsey-Cass-Koopmans model by starting from a social planner's problem without bothering to explore the sophisticated flow budget constraint which a representative consumer is facing in a decentralized economy.

The answer to this question is, yes, but unfortunately, quite restricted.

From the first look, one may be tempted to feel that there might be something to do with the Theorems of Welfare Economics and would wonder whether we can say anything with the Theorems. Sure, here for the current settings the decentralied economy is featured by a competitive (Walrasian) equilibrium, and the allocation proposed by the social planner is Pareto optimal. The First Theorem of Welfare Economics tells us that if the equilibrium is Walrasian, then the allocation is Pareto optimal (which *sounds* that the decentralized equilibrium allocation may coincide the social planner's solution); and the Second Theorem of Welfare Economics tells us that if an allocation is Pareto optimal (in which each agent holds a positive amount of each good, and the preferences are convex, continuous and monotonic), then such allocation is a Walrasian equilibrium given some initial endowments (which *sounds* that the social planner's solution can be replicated by a competitive equilibrium).

Then why can't we say that these two approaches are equivalent? Alas, the Theorems of Welfare Economics (at least as in Mas-Colell *et al.*) assume that the commodity space is *finite*, however, our infinitely living agent is maximizing a sequence of $\{c_t\}_{t=0}^{+\infty}$ — suggesting that the commodity space here is *infinite*. Therefore, under the settings different from standard Ramsey, such infinite commodity space may bring us new troubles and the Theorems are likely to fail (one can have a look at the proofs of the Theorems and think why this happens).

So one has to be careful when the settings of the problems at hand deviate from the standard ones, and think it over how the parameters affect the decisions (i.e. through individual's optimal behavior, or central planning). Basically, whenever the household's budget constraint or utility function differs in the right ways from the social budget constraint or the central planner's preferences, there can be a divergence between the solutions of the two approaches. Just name a few occasions:

- There are externalities in the behavior of individual households (e.g. our exercise attached to the class of dynamic programming);
- There are idiosyncratic risks among the consumers but no aggregate risks for the economy as a whole;
- There are distortionary taxes (e.g. Problem 2 from Problem Set 2);
- The consumers have different time horizons from the central planner's. For example, when the representative consumer has a finite life span while the central planner is eternal (e.g. Problem 5 and 6 from Problem Set 2), or the central planner has a generally short tenure in her office as in some political economic models

and so on.

3 Nothing But a Jump

As we have already seen before, phase diagram provides us a very powerful tool for charaterizing the convergence path. Moreover, it is also a very useful method to analyse the economy's qualitative response to the policy change.

Since the decisions on the control variables, hence the change of stock variables, are normally made by the forward-looking agents (sometimes people call "agents with rational expectations", or "agents with perfect foresight", etc.), it's important in the first place to specify whether the change is expected or unexpected. The reason is pretty intuitive: If a policy change is expected, the agent may modify her behavior before the change actually takes place. In the section that follows, we show how agents respond to unexpected / expected policy changes in a purely qualitative way. In addition, to make our arguments simpler (and without loss of generality) from now on we assume that the economy is already on the balanced growth path (i.e. steady state) when the (prospective) policy change is known, and the policy change will sustain permenantly (or at least for a sufficiently long time into the future) after it realizes.

3.1 Unexpected Policy Change

If a policy change is unexpected, then the system dynamics change immediately when the new policy comes in, i.e. the phase diagram (hence the corresponding saddle path) may change under the new system parameters. Suppose that the agent still stays in the old steady state, she would end up with disaster (i.e. hitting the "walls") under the new system dynamics. To avoid this, the only thing she can do at this time is to jump (via adjusting the control variables) onto the new saddle path which leads her to the new steady state.

To make it visible, an example is given as following: the settings are taken from Problem 1 of Problem Set 2, and we introduce an unexpected increase in depreciation rate δ.

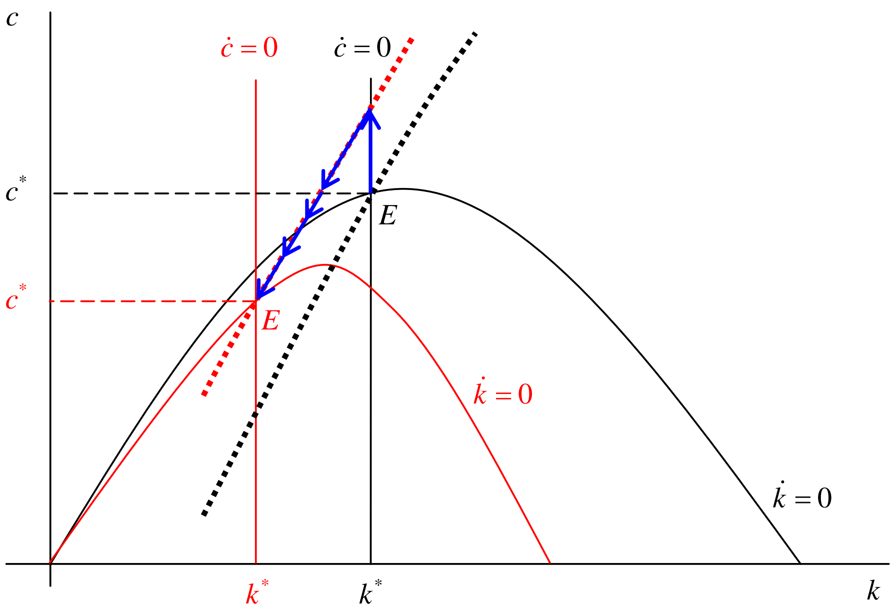

Fig. 1. An Unexpected Increase in δ

The transition process is shown in Figure 1. The phase diagram under the old parameters is plotted in black lines, and the phase diagram under the new parameters is plotted in red lines. Now when the policy comes in, the old steady state (black E) is off the new saddle path. If the agent still stays here she would end up hitting k-axis in finite time, driven by the new system dynamics. This would violate the transversality condition, and a rational agent should avoid such a disaster and must arrive at the new saddle path when the policy realizes. Remember that k, as a state variable, cannot change discontinuously, therefore the only thing that the agent can do is to make an immediate change in c — A jump in c dimension as the blue arrow shows, and then converge to the new steady state (red E).

Figure 2 and 3 plot the results from a numerical simulation [1]. To make the computation simpler, we express the instantaneous utility function as $u(t) = \frac{1}{\beta}c(t)^\beta$ and let $\beta \to 0$ (this gives us a log-utility function). The initial values for the parameters are

$$\alpha = 0.3,$$
$$n = 0,$$
$$\delta = 0.03,$$
$$\rho = 0.05.$$

And the old stead state is

$$c^* = 1.5638,$$

[1] Since the computer is not able to deal with continuous functions, we discretize the time horizon into many periods.

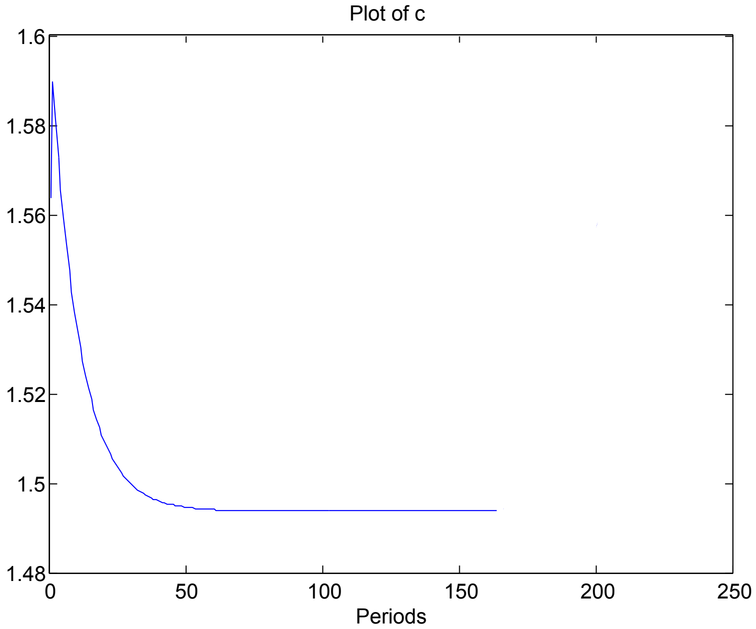

Fig. 2. The Transition Process of Consumption

$k^* = 6.6076.$

Now an unexpected policy is introduced such that δ becomes 20% higher. Figure 2 and 3 show how c and k evolve after the policy shock, which are exactly the same as we predicted in Figure 1.

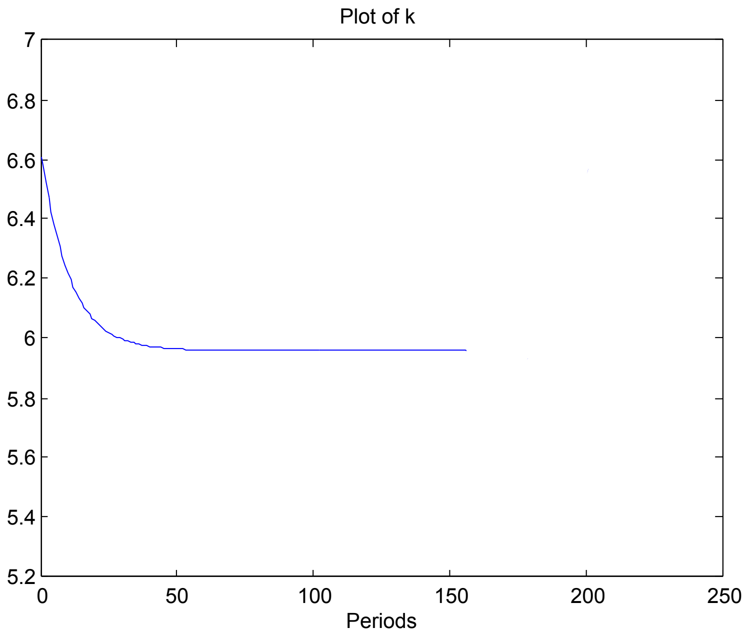

Fig. 3. The Transition Process of Capital Stock

Then people may wonder why the agent has to change c, the consumption level, *immediately* after the policy is known. Could it be justified that c is changed some time later? Suppose that there is an alternative scheme, as shown in FIGURE 4: When the policy is known at t_0, the agent still stays at the old steady state (black c^*). After some time, at t_1 the agent jumps onto the new saddle path and then moves towards the new steady state (red c^*) [2]. Could the agent be better off if she follows such a scheme?

The answer is no. We prove this by contradiction.

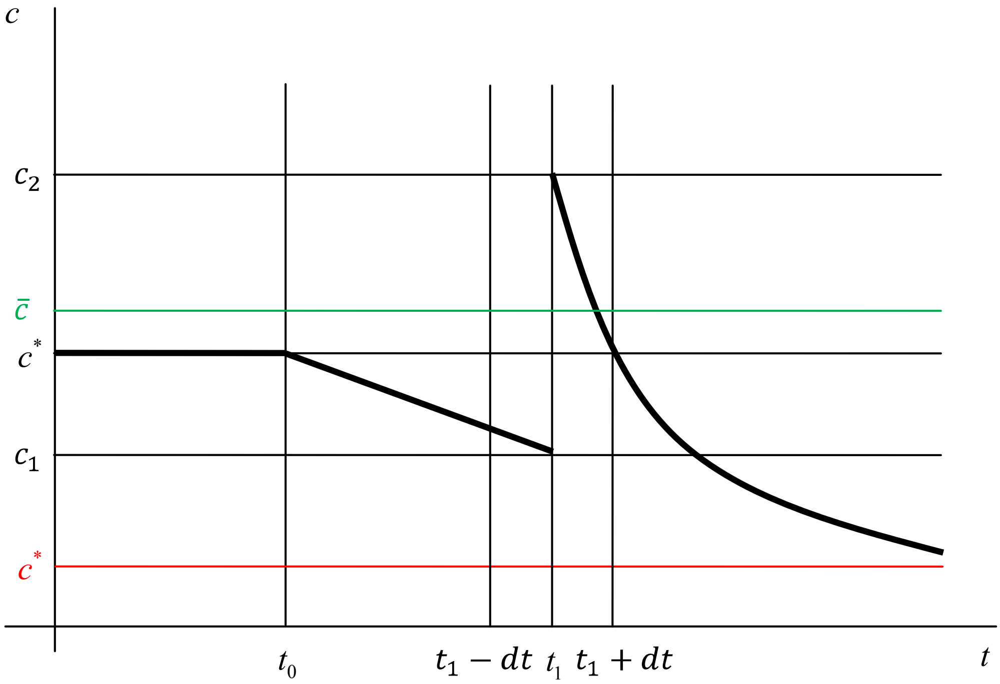

Fig. 4. AN ALTERNATIVE CONSUMPTION SCHEME

Suppose that such scheme is an optimal consumption path for the agent, in which the consumption level jumps from c_1 to c_2 at time t_1. Then we take a sufficiently small dt neighborhood around t_1, and the agent's utility gain in $[t_1 - dt, t_1 + dt]$ can be expressed as

$$u(t_1 - dt, t_1 + dt) = u(c_1)dt + u(c_2)dt.$$

Now we take the average $\bar{c} = \frac{c_1+c_2}{2}$. Then given that $u(\cdot)$ is strictly concave, by Jensen's inequality

$$\begin{aligned}
&\frac{u(c_1) + u(c_2)}{2} dt < u\left(\frac{c_1 + c_2}{2}\right) dt, \\
&u(c_1)dt + u(c_2)dt < u\left(\bar{c}\right)(2dt), \\
&u(t_1 - dt, t_1 + dt) < u\left(\bar{c}\right)(2dt).
\end{aligned}$$

[2] Note that she cannot really stick to black c^* after t_0 — The new system dynamics will drive c downwards.

This means that if the consumption level jumps from c_1 to c_2 at time t_1 one can always find a better scheme with a constant $\overline{c}$ in $[t_1 - dt, t_1 + dt]$, yielding higher utility for the agent. But this contradicts the assumption that it's optimal for the agent to jump from c_1 to c_2 at time t_1. Therefore, any consumption path containing a jump (or jumps) after t_0 cannot be optimal. Reversely speaking, the optimal consumption path only allows a jump when the policy is known (i.e. at t_0) — the only time point that the agent is not able to smooth her consumption levels before and after t_0.

3.2 *Expected Policy Change*

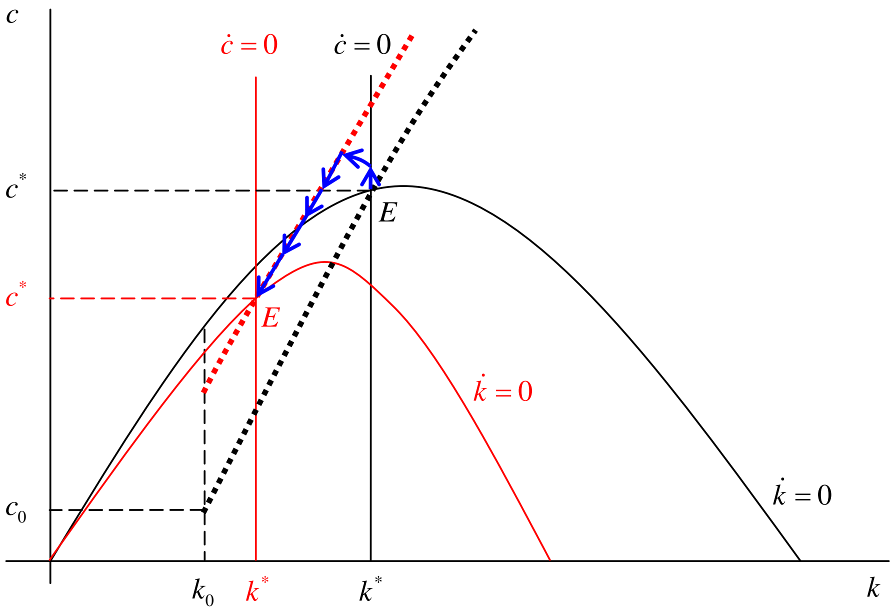

Fig. 5. A Foreseeable Increase in n

It would be tricky if the policy change is expected, because the agent with perfect foresight may find it optimal to smooth her decisions before the change realizes. To make such situation clear, think about the question in Problem 6 from Problem Set 1: At t_0 news comes that the growth rate in labor supply will increase permenantly after t_1.

Figure 5 shows the phase diagrams under old (lines in black) and new (lines in red) parameters. Obviously the agent will end up at the new steady state, but the question is: How?

Here is the qualitative reasoning:

- Surely the agent cannot jump directly from the old steady state (black E) to the new one (red E), because k cannot change discontinuously;
- Then at t_1 the agent should be somewhere along the saddle path, then move towards the new steady state. If she is not on the saddle path at t_1, as we argued before, she would end up with disaster;

- Then the agent should move to reach the new saddle path during $[t_0, t_1]$. As we proved several paragraphs before, if the agent is behaving optimally, no jump (i.e. a discontinuous change in c, for simplicity) is allowed after t_0. What's more, if the agent wants to jump, then t_0 is the only chance;
- Should she jump at t_0? Yes, otherwise she would not move before t_1 at all.

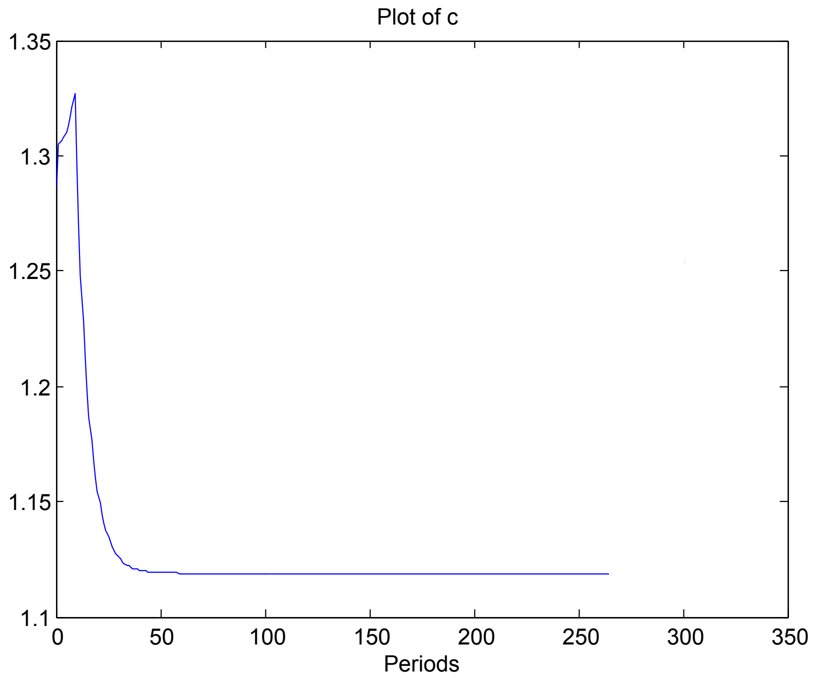

Fig. 6. The Transition Process of Consumption with Perfect Foresight

The transition process is summerized in Figure 5. When the news comes at t_0 the agent jumps off from the old steady state. But for any time before t_1 the system is still governed by the old dynamics, therefore the growth path is driven in the northwestern direction. When the policy finally realizes at t_1 the growth path will exactly hit the new saddle path, leading the agent towards the new steady state.

Again, we visualize the process via a numerical simulation. The parameters are similar as before except that the growth rate of labor supply is positive

$$\alpha = 0.3,$$
$$\beta \to 0,$$
$$n = 0.03,$$
$$\delta = 0.03,$$
$$\rho = 0.05.$$

And the old stead state is

$$c^* = 1.2857,$$

$k^* = 4.1925.$

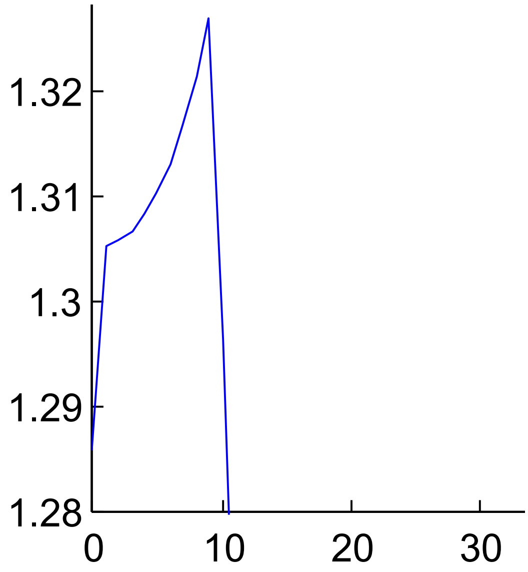

Fig. 7. The Transition Process of Consumption Before The Shock Realizes under Perfect Foresight

Now news comes at $t_0 = 0$ that n doubles permanently from $t_1 = 10$ onwards. Figure 6 shows how c evolves after t_0, and Figure 7 shows the details how c changes before t_1.

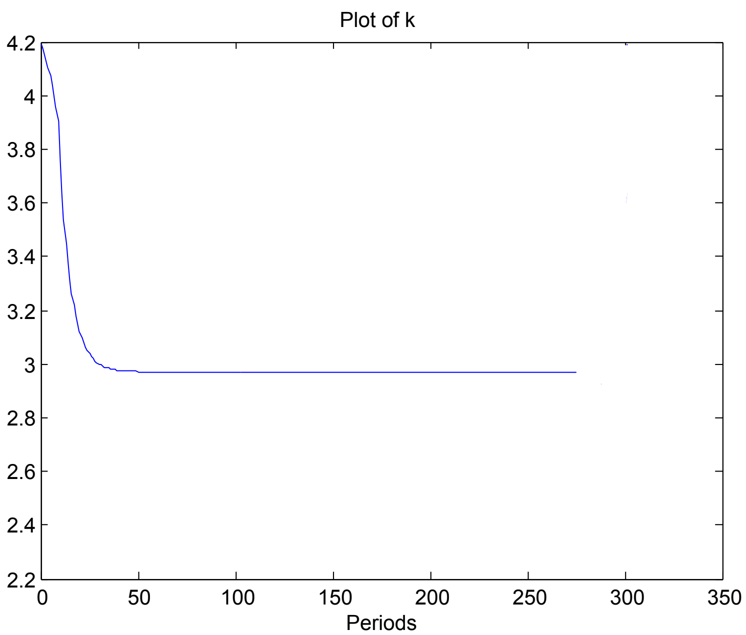

Fig. 8. The Transition Process of Capital Stock with Perfect Foresight

Figure 8 shows that k decreases all the way, but there is a turning point at t_1 because its growth rate changes due to the change in c. Both figures suggest that we predict a right pattern via Figure 5.

4 Overlapping Generation Model

Now we come to the overlapping generation model, which is another major modelling framework in macroeconomic studies. The key difference between the overlapping generation model and the Ramsey-Cass-Koopmans model is that there is a *built-in* imperfection: rather than there being a fixed number of infinitely living households, in overlapping generation models new individuals are continually being born and old ones are continually dying. The direct implication of this assumption is that the social planner's planning horizon may differ from the representative agent's life span. And we will see this leads to many interesting results.

4.1 Basic Settings

To make it generic we maintain the same two sectors, households and firms, as we did in Ramsey problem. With turnover in the population, it is simpler to assume that the time is discrete from now on.

4.1.1 Households

Each person in this economy lives for two periods. The group that is born at t is called generation t, and during period t the young (generation t) overlaps with the old (generation $t-1$). The population of generation t is L_t, and L grows at a constant rate n overtime, i.e. $L_t = (1+n)L_{t-1}$. In each period only two generations are alive. One representative agent's activities in each period:

- First period: working for earning a wage w_t, consuming c_t^y, saving s_t;
- Second period: retiring, and consuming c_{t+1}^o from the return of the savings $(1+r_{t+1})s_t$.

Each person at t gains utility from young and old-age consumption according to

$$u_t = u\left(c_t^y, c_{t+1}^o\right),$$

in which the utility function has all the properties we assumed in the first chapter of the class notes.

Then in this economy a representative agent's problem can be expressed as

$$\begin{aligned}
\max_{c_t^y, c_{t+1}^o} \; & u_t = u\left(c_t^y, c_{t+1}^o\right), \\
s.t. \; & c_t^y + s_t \leq w_t, \\
& c_{t+1}^o \leq (1+r_{t+1})s_t.
\end{aligned}$$

In optimum the two inequality constraints must be binding, and they can be merged into one

$$c_{t+1}^{o} = (1 + r_{t+1})(w_t - c_t^{y}).$$

Set up Lagrangian

$$\mathscr{L} = u\,(c_t^{y}, c_{t+1}^{o}) + \lambda\,[(1 + r_{t+1})(w_t - c_t^{y}) - c_{t+1}^{o}]\,.$$

First order conditions are

$$\frac{\partial \mathscr{L}}{\partial c_t^{y}} = \frac{\partial u\left(c_t^{y}, c_{t+1}^{o}\right)}{\partial c_t^{y}} - \lambda(1 + r_{t+1}) = 0,$$
$$\frac{\partial \mathscr{L}}{\partial c_{t+1}^{o}} = \frac{\partial u\left(c_t^{y}, c_{t+1}^{o}\right)}{\partial c_{t+1}^{o}} - \lambda = 0.$$

Eliminate λ and get the Euler equation

$$-\frac{\partial u\left(c_t^{y}, c_{t+1}^{o}\right)}{\partial c_t^{y}} + (1 + r_{t+1})\frac{\partial u\left(c_t^{y}, c_{t+1}^{o}\right)}{\partial c_{t+1}^{o}} = 0, \tag{9}$$

meaning that the marginal utility from consumption today must equal marginal utility tomorrow, corrected by the interest rate. If we insert the budget constraints into it, we can define a function relating the saving rate to the wage income and the interest rate

$$s_t = s(w_t, r_{t+1}). \tag{10}$$

To see the impacts of w_t and r_{t+1} on s_t, first we differentiate the implicit function (9) with respect to w_t

$$\left[\frac{\partial^2 u\left(c_t^{y}, c_{t+1}^{o}\right)}{\partial\,(c_t^{y})^2}\frac{\partial c_t^{y}}{\partial w_t} + \frac{\partial^2 u\left(c_t^{y}, c_{t+1}^{o}\right)}{\partial c_t^{y}\partial c_{t+1}^{o}}\frac{\partial c_{t+1}^{o}}{\partial s_t}\frac{\partial s_t}{\partial w_t}\right]$$
$$= (1 + r_{t+1})\left[\frac{\partial^2 u\left(c_t^{y}, c_{t+1}^{o}\right)}{\partial c_{t+1}^{o}\partial c_t^{y}}\frac{\partial c_t^{y}}{\partial w_t} + \frac{\partial^2 u\left(c_t^{y}, c_{t+1}^{o}\right)}{\partial\left(c_{t+1}^{o}\right)^2}\frac{\partial c_{t+1}^{o}}{\partial s_t}\frac{\partial s_t}{\partial w_t}\right].$$

From budget constraints one can see that

$$\frac{\partial c_t^{y}}{\partial w_t} = 1 - \frac{\partial s_t}{\partial w_t},$$

$$\frac{\partial c^o_{t+1}}{\partial s_t} = 1 + r_{t+1}.$$

Combine these two with the last equation, one can see that

$$\frac{\partial s_t}{\partial w_t} = \frac{\frac{\partial^2 u(c^y_t, c^o_{t+1})}{\partial (c^y_t)^2} - (1 + r_{t+1})\frac{\partial^2 u(c^y_t, c^o_{t+1})}{\partial c^o_{t+1}\partial c^y_t}}{(1 + r_{t+1})^2 \frac{\partial^2 u(c^y_t, c^o_{t+1})}{\partial (c^o_{t+1})^2} - 2(1 + r_{t+1})\frac{\partial^2 u(c^y_t, c^o_{t+1})}{\partial c^y_t \partial c^o_{t+1}} + \frac{\partial^2 u(c^y_t, c^o_{t+1})}{\partial (c^y_t)^2}}.$$

Then differentiate the implicit function (9) with respect to r_{t+1}

$$\begin{aligned}
&\left[\frac{\partial^2 u\left(c^y_t, c^o_{t+1}\right)}{\partial \left(c^y_t\right)^2}\frac{\partial c^y_t}{\partial s_t}\frac{\partial s_t}{\partial r_{t+1}} + \frac{\partial^2 u\left(c^y_t, c^o_{t+1}\right)}{\partial c^y_t \partial c^o_{t+1}}\left(s_t + (1 + r_{t+1})\frac{\partial s_t}{\partial r_{t+1}}\right)\right] \\
&= (1 + r_{t+1})\left[\frac{\partial^2 u\left(c^y_t, c^o_{t+1}\right)}{\partial c^o_{t+1}\partial c^y_t}\frac{\partial c^y_t}{\partial s_t}\frac{\partial s_t}{\partial r_{t+1}} + \frac{\partial^2 u\left(c^y_t, c^o_{t+1}\right)}{\partial \left(c^o_{t+1}\right)^2}\right. \\
&\quad \left.\cdot\left(s_t + (1 + r_{t+1})\frac{\partial s_t}{\partial r_{t+1}}\right)\right] + \frac{\partial u\left(c^y_t, c^o_{t+1}\right)}{\partial c^o_{t+1}}.
\end{aligned}$$

From budget constraints one can see that

$$\frac{\partial c^y_t}{\partial s_t} = -1,$$

insert this into the last equation and we get

$$\frac{\partial s_t}{\partial r_{t+1}} = \frac{s_t \frac{\partial^2 u(c^y_t, c^o_{t+1})}{\partial c^y_t \partial c^o_{t+1}} - \frac{\partial u(c^y_t, c^o_{t+1})}{\partial c^o_{t+1}} - s_t(1 + r_{t+1})\frac{\partial^2 u(c^y_t, c^o_{t+1})}{\partial (c^o_{t+1})^2}}{(1 + r_{t+1})^2 \frac{\partial^2 u(c^y_t, c^o_{t+1})}{\partial (c^o_{t+1})^2} - 2(1 + r_{t+1})\frac{\partial^2 u(c^y_t, c^o_{t+1})}{\partial c^y_t \partial c^o_{t+1}} + \frac{\partial^2 u(c^y_t, c^o_{t+1})}{\partial (c^y_t)^2}}.$$

Now we try to determine the signs of $\frac{\partial s_t}{\partial w_t}$ and $\frac{\partial s_t}{\partial r_{t+1}}$. Known from the strict concavity of the utility function that

$$\frac{\partial^2 u\left(c^y_t, c^o_{t+1}\right)}{\partial \left(c^y_t\right)^2} < 0,$$
$$\frac{\partial^2 u\left(c^y_t, c^o_{t+1}\right)}{\partial \left(c^o_{t+1}\right)^2} < 0,$$

and from the normality of the utility function that

$$\frac{\partial^2 u\left(c_t^y, c_{t+1}^o\right)}{\partial c_t^y \partial c_{t+1}^o} \geq 0,$$

the denominator of $\frac{\partial s_t}{\partial w_t}$ shows that

$$D_t = (1 + r_{t+1})^2 \frac{\partial^2 u\left(c_t^y, c_{t+1}^o\right)}{\partial \left(c_{t+1}^o\right)^2} - 2(1 + r_{t+1}) \frac{\partial^2 u\left(c_t^y, c_{t+1}^o\right)}{\partial c_t^y \partial c_{t+1}^o} + \frac{\partial^2 u\left(c_t^y, c_{t+1}^o\right)}{\partial \left(c_t^y\right)^2} < 0,$$

and the numerator implies that

$$\frac{\partial^2 u\left(c_t^y, c_{t+1}^o\right)}{\partial \left(c_t^y\right)^2} - (1 + r_{t+1}) \frac{\partial^2 u\left(c_t^y, c_{t+1}^o\right)}{\partial c_{t+1}^o \partial c_t^y} < 0.$$

And then we can see that

$$\frac{\partial s_t}{\partial w_t} > 0$$

implying that the income effect dominates.

However, the joint effects of substitution and income effects leave the sign of $\frac{\partial s_t}{\partial r_{t+1}}$ undetermined.

4.1.2 Production

Firms have the typical neo-classical production function (suppose that there is no technological progress)

$$Y_t = F\left(K_t, L_t\right).$$

Write in per-capita form

$$y_t = f\left(k_t\right),$$

in which $y_t = \frac{Y_t}{L_t}$ and $k_t = \frac{K_t}{L_t}$.

Firms are competitive such that

$$r(k_t) = f'(k_t) - \delta, \tag{11}$$
$$w(k_t) = f(k_t) - k_t f'(k_t), \tag{12}$$

which we have seen many times.

4.2 Equilibrium

In the next step we put the two sectors together and see how the allocation looks like in the equilibrium.

4.2.1 Equilibrium Conditions

The equilibrium can be characterized as a sequence of decisions $\{c_t^y, c_t^o, s_t, r_t, w_t, K_{t+1}, L_t, Y_t\}_{t=0}^{+\infty}$, such that the following conditions have to hold:

- Utility maximization by households, characterized by (9) or (10);
- Profit maximization by firms, characterized by (11) and (12);
- Market clearing
  - From the side of households

$$\begin{aligned} K_{t+1} - K_t &= w_t L_t + r_t K_t - c_t^y L_t - c_{t-1}^o L_{t-1} \\ &= w_t L_t + r_t K_t - (w_t - s_t) L_t - (1 + r_t) s_{t-1} L_{t-1}, \\ K_{t+1} &= s_t L_t + (1 + r_t)(K_t - s_{t-1} L_{t-1}). \end{aligned}$$

Suppose that we start from $t = 1$ with initial capital stock K_1 owned by population L_0 who are old (i.e. generation 0) in this period. These people consume $c_1^o L_0 = (1 + r_1) K_1$. Apply this to the equation above and get

$$\begin{aligned} K_2 &= s_1 L_1 + (1 + r_1) K_1 - (1 + r_1) s_0 L_0 \\ &= s_1 L_1 + \underbrace{c_1^o L_0 - (1 + r_1) s_0 L_0}_{=0} \\ &= s_1 L_1, \end{aligned}$$

in which $c_1^o L_0 - (1 + r_1) s_0 L_0 = 0$ comes from the budget constraint. Apply this result to the expression of K_{t+1} and we find that

$$K_{t+1} = s_t L_t, \forall t \geq 1.$$

Rewrite it in per capita form

$$\begin{aligned} k_{t+1} L_{t+1} &= s_t L_t \\ s_t &= (1 + n) k_{t+1}. \end{aligned}$$

Using (11) and (12) we can capture all the conditions in a reduced form

$$s_t\left[w(k_t), r(k_{t+1})\right] = (1+n)k_{t+1}. \tag{13}$$

· From the side of firms

$$K_{t+1} - K_t = F(K_t, L_t) - c_t^y L_t - c_t^o L_{t-1} - \delta K_t, \tag{14}$$

rearrange to get

$$K_{t+1} = K_t + (r_t + \delta)K_t + w_t L_t - c_t^y L_t - c_t^o L_{t-1} - \delta K_t,$$
$$K_{t+1} = s_t L_t + (1 + r_t)K_t - c_t^o L_{t-1}.$$

Then it's trivial to get the same result

$$s_t\left[w(k_t), r(k_{t+1})\right] = (1+n)k_{t+1}$$

using the fact that $c_t^o = (1 + r_t)s_{t-1}$.

4.2.2 Steady State

From (13) the equilibrium path is totally determined by k_t and k_{t+1}. The steady state is defined such that the capital intensity is constant over time, i.e. $k_t = k_{t+1} = k^*$. Therefore the steady state value k^* can be solved by setting

$$s\left[w(k^*), r(k^*)\right] = (1+n)k^*. \tag{15}$$

4.3 Example

Suppose that the representative agent's utility function takes the form

$$u\left(c_t^y, c_{t+1}^o\right) = \ln c_t^y + \beta \ln c_{t+1}^o, \beta \in (0,1),$$

and the firms take Cobb-Douglas technology $F(K_t, L_t) = K_t^\alpha L_t^{1-\alpha}$. You may find that

- s_t is independent on r_{t+1}, i.e. $s_t = s[w(k_t)]$;
- The wage rate is a constant fraction of aggregate output, i.e. $w_t = (1-\alpha)y_t$;
- The system dynamics are captured by

$$k_{t+1} = \frac{\beta}{1+\beta}\frac{1-\alpha}{1+n}k_t^\alpha, \tag{16}$$

  and it's easily seen from Figure 9 that the curve $k_{t+1} = \phi(k_t)$ from (16) has a single cross with 45 degree line, meaning that there exists a unique steady state in which $k_{t+1} = k_t = k^*$ and $0 < k^* < +\infty$. The system dynamics are also depicted for two arbitrary initial values;

- The corresponding steady state capital intensity is

$$k^* = \left(\frac{\beta}{1+\beta}\frac{1-\alpha}{1+n}\right)^{\frac{1}{1-\alpha}}.$$

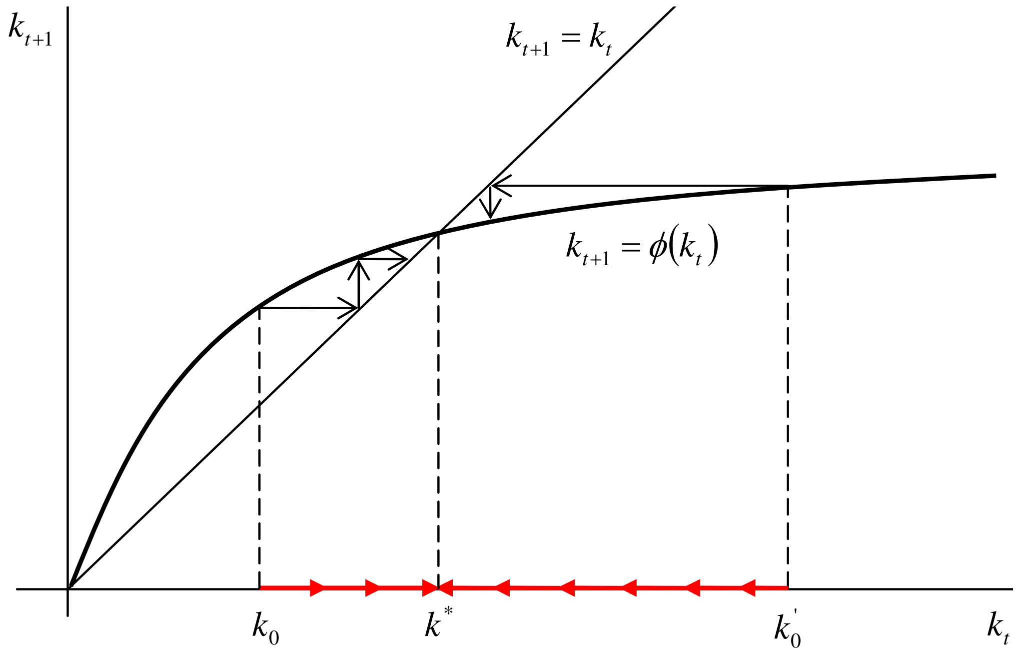

Fig. 9. The Steady State

4.4 Discussions

Albeit many differences in the settings, the overlapping generation model so far generated quite similar convergence as Solow-Swan model. However, the differences in settings do have profound impacts on the outcomes, and the seemingly similar results may imply totally different notions.

4.4.1 The Steady State

Remember that in Solow-Swan model the unique, non-trivial (i.e. $k^* \neq 0$) steady state is ensured by the single cross between the scaled neoclassical production function $sf\left(\hat{k}\right)$ and $(n + \delta + g)\hat{k}$ line. Does the same principle apply for seeking the steady state of overlapping generation models, as in Figure 9?

In general, the relation $k_{t+1} = \phi(k_t)$ is pinned down by (13), which is an implicit, non-linear function of k_t and k_{t+1}. Without knowing exact forms of utility function and production function, we have almost no idea about the properties of this implicit function. Therefore, we must carefully examine the existence and uniqueness of the steady state!

However, it would be quite difficult to get any definite result directly from equation (13). Therefore, we start from an alternative approach, by analyzing $w(k_t)$ function.

Notice that $\forall t$ the budget constraint for the young generation, $c_t^y + s_t \le w_t$, has to hold. Given that $c_t^y \ge 0$, $s_t \le w_t$. Combining with equation (13), one can get $(1+n)k_{t+1} \le w(k_t)$, i.e. $k_{t+1} \le \frac{w(k_t)}{1+n}$. Plot $\frac{w(k_t)}{1+n}$ curve as FIGURE 10, and $k_{t+1} = \frac{s_t}{1+n}$ curve should be no higher since $s_t \le w_t(k_t)$. However, if $\frac{w(k_t)}{1+n} \le k_t$, $\forall k_t$ as in FIGURE 10, the trivial outcome $k^* = 0$ is the only steady state.

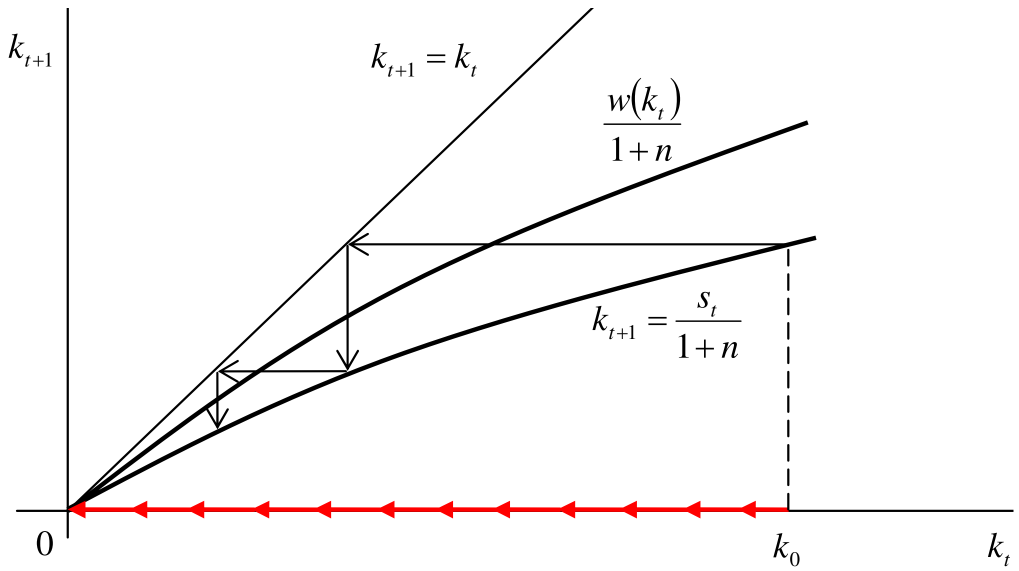

Fig. 10. THE TRIVIAL STEADY STATE

To exlude such case, we have to impose more restrictions. From FIGURE 11, one can see that to ensure the existence of a non-trivial steady state, such that $k_{t+1} = \frac{s_t}{1+n}$ curve has a cross with 45 degree line at $k^* > 0$ as curve A shows, the slope of $\frac{w(k_t)}{1+n}$ must exceeds 1 at $k_t \to 0$. However, the vice versa isn't true because the case $k_{t+1} = \frac{s_t}{1+n} \le k_t$, $\forall k_t$ *may* still exist even if $\lim_{k_t \to 0} \frac{1}{1+n} \frac{dw(k_t)}{dk_t} > 1$ holds, as curve B. Therefore, with $w(k_t)$ being defined by (12), one can easily find the *necessary* condition for the existence of a non-trivial steady state.

Proposition 4.1 *In the overlapping generation model, there exists a non-trivial steady state* $k^* > 0$*, such that* $\lim_{t\to+\infty} k_t = k^*$*,* $\forall k_0 > 0$ *only if*

$$\lim_{k_t \to 0} [-k_t f''(k_t)] > 1 + n.$$

Proof When $\lim_{t\to+\infty} k_t = k^* > 0$, $\forall k_0 > 0$, then $k_{t+1} > k_t$, $\forall k_0 \in (0, k^*)$. Therefore

$$
\begin{aligned}
k_t < k_{t+1} \leq \frac{w(k_t)}{1+n} &= \frac{f(k_t) - k_t f'(k_t)}{1+n}, \forall k_0 \in (0, k^*), \\
1+n &< \frac{f(k_t) - k_t f'(k_t)}{k_t}.
\end{aligned}
$$

Applying L'Hôpital's Rule,

$$
\lim_{k_t \to 0} \frac{f(k_t) - k_t f'(k_t)}{k_t} = \lim_{k_t \to 0} [-k_t f''(k_t)] > 1 + n.
$$

□

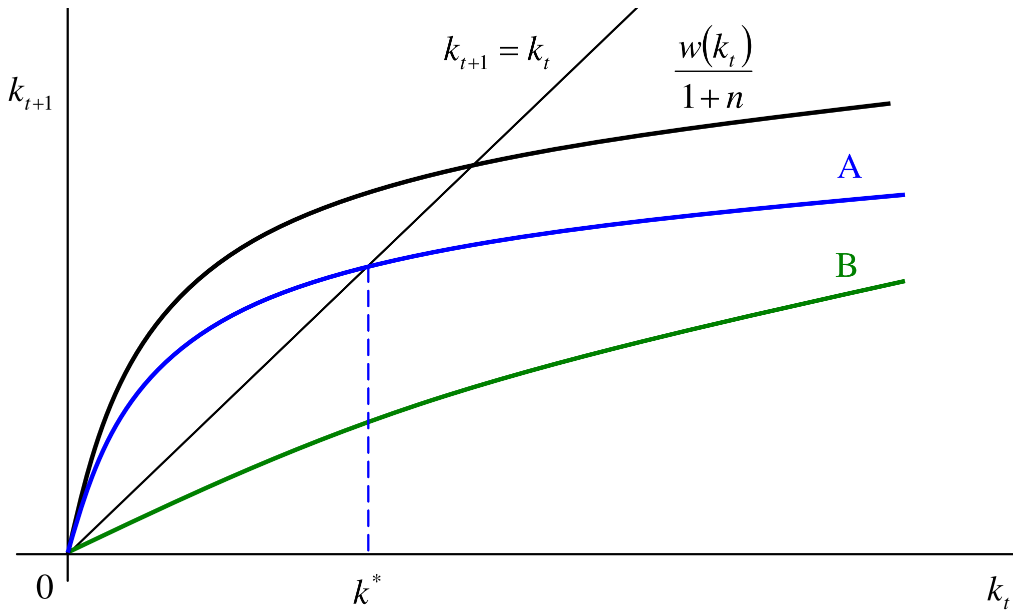

Fig. 11. The Non-Trivial Steady State: A Necessary Condition

Notice the condition is more strict than the Inada conditions. That is, Inada conditions are no longer sufficient to ensure non-trivial steady states in overlapping generation models.

Proposition 4.1 only deals with the case in which a non-trivial steady state exists, however, we have yet no idea whether the steady state is unique / stable. For example, Figure 12 presents one case with multiple non-trivial steady states, two of which being global stable. Therefore, if we want to see the neat, globally stable unique equilibrium such as in Solow-Swan model, we simply have to impose more restrictions (and, unfortunately, we have to confront with the implicit function in (13)). Some *sufficient* conditions are suggested in Proposition 4.2.

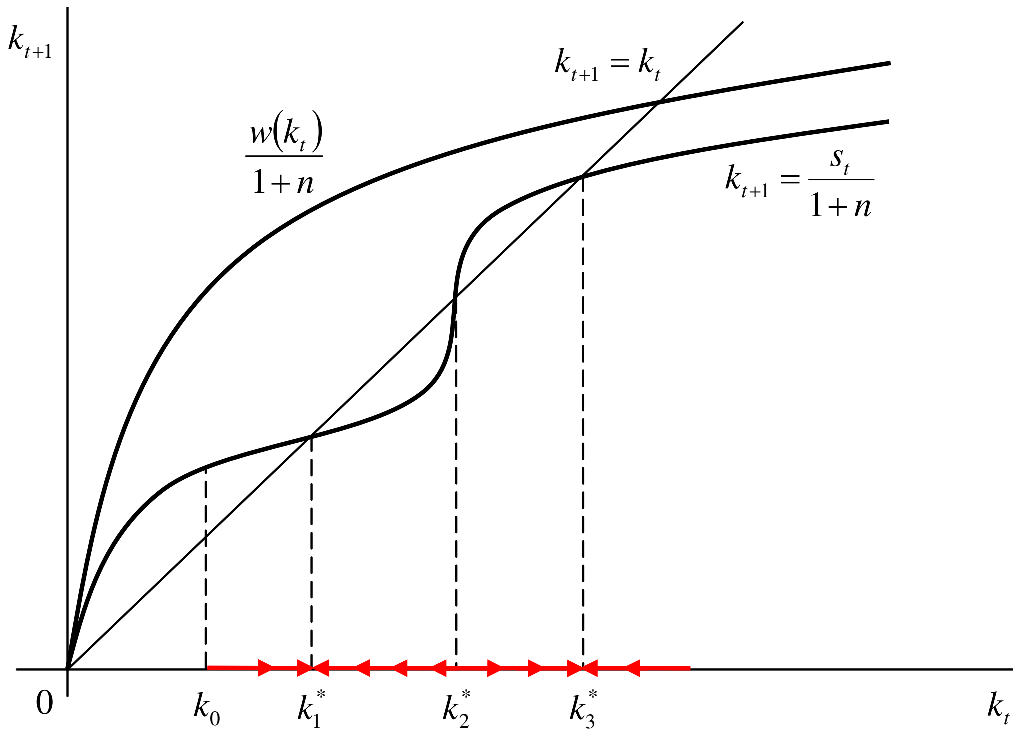

Fig. 12. Multiple Equilibria ("Poverty Trap")

Proposition 4.2 *In the overlapping generation model, there exists a unique, globally stable, and non-trivial steady state* $\forall k_0 > 0$ *if*

(1)

$$\lim_{k_t \to 0} \phi'(k_t) = \frac{-\frac{\partial s_t}{\partial w_t} k_t f''(k_t)}{1 + n - \frac{\partial s_t}{\partial r_{t+1}} f''(k_{t+1})} > 1, \text{ in which } k_{t+1} = \phi(k_t); \tag{17}$$

(2)

$$\lim_{k_t \to +\infty} f'(k_t) = 0; \tag{18}$$

(3)

$$\phi'(k_t) \geq 0, \forall k_t > 0; \tag{19}$$

(4)

$$\phi''(k_t) < 0, \forall k_t > 0; \tag{20}$$

(5)

$$\frac{\partial s_t}{\partial r_{t+1}} \geq 0, \forall (w_t, r_{t+1}) \geq 0. \tag{21}$$

Proof (Sketch) Take total differentiation of equation (13),

$$\frac{\partial s_t}{\partial w(k_t)}\frac{dw(k_t)}{dk_t}dk_t + \frac{\partial s_t}{\partial r(k_{t+1})}\frac{dr(k_{t+1})}{dk_{t+1}}dk_{t+1} - (1+n)dk_{t+1} = 0,$$

$$\frac{\partial s_t}{\partial w(k_t)}\left[f'(k_t) - f'(k_t) - kf''(k_t)\right]dk_t + \frac{\partial s_t}{\partial r(k_{t+1})}f''(k_{t+1})\,dk_{t+1} - (1+n)dk_{t+1} = 0,$$

solve to get

$$\frac{dk_{t+1}}{dk_t} = \frac{-\frac{\partial s_t}{\partial w_t}k_t f''(k_t)}{1+n-\frac{\partial s_t}{\partial r_{t+1}}f''(k_{t+1})} = \phi'(k_t),$$

which is the $\phi'(k_t)$ part of condition (17).

Like curve A in Figure 11, it's natural to see that the following requirements can ensure a unique, globally stable, and non-trivial steady state $k^* > 0$:

(a) $\phi(k_t)$ is a single-valued function;
(b) The function $\phi(k_t)$ is increasing and strictly concave in k_t;
(c) $\lim_{k_t \to 0} \phi'(k_t) > 1$;
(d) $\phi(k_t)$ crosses 45 degree line at $0 < k^* < +\infty$.

Condition (21) is sufficient for **(a)**. Condition (17) is identical to **(c)**, (19) plus (20) imply **(b)**, and **(b)** plus (18) imply **(d)** (notice that (18) implies that $\lim_{k_t \to +\infty} \phi'(k_t) = 0$). □

Readers should keep in mind that our effort on working out the two propositions above, guaranteeing certain types of neat steady states, doesn't mean that the other "maverick"ones are less interesting. In fact, the rich set of these "outliers"is of increasing importance for economists to better understand the complex real world, for example, the *poverty trap* hidden in Figure 12 (How if a developing country starts its economic growth from k_0? See Exercise 7.7), the *sinuous growth path* implied by the case in Figure 13, and even, chaos.

4.4.2 The Golden Rule

As emphasized before, the representative agent in this economy has a finite life horizon, whereas the economy lives on forever. Therefore, the agent's decision might be inefficient from the social planner's point of view. To see this, we start from comparing the equilibrium allocation in this model with the outcome under the golden rule.

Rewrite the law of motion (14) here

$$\begin{aligned} K_{t+1} - K_t &= F(K_t, L_t) - c_t^y L_t - c_t^o L_{t-1} - \delta K_t \\ &= F(K_t, L_t) - C_t - \delta K_t \end{aligned}$$

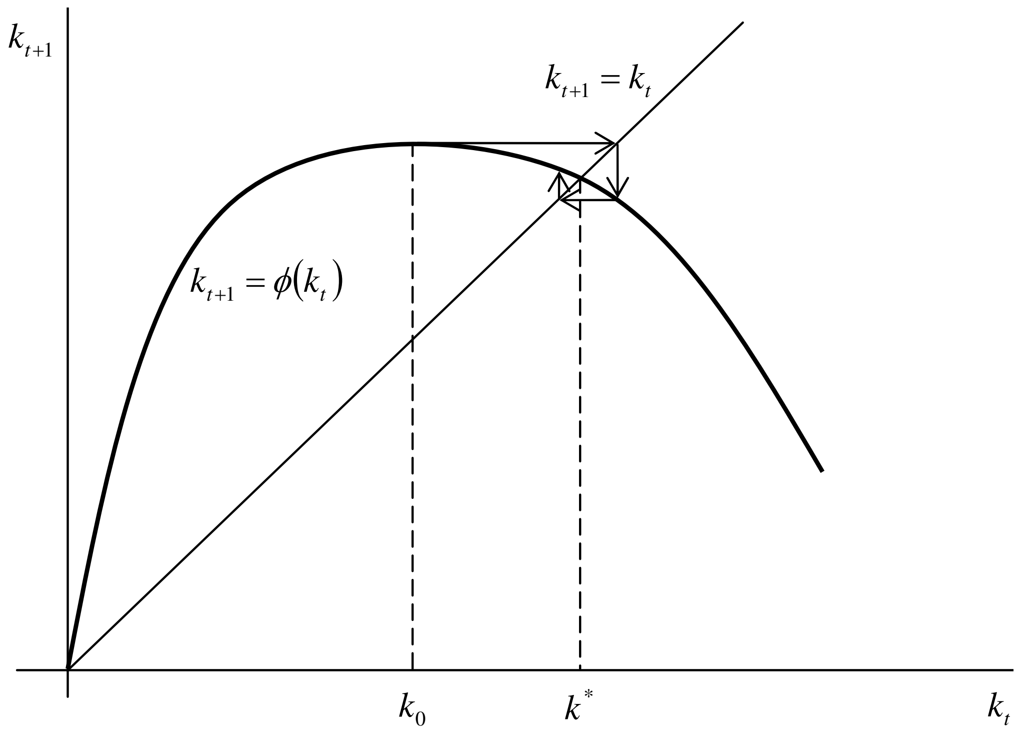

Fig. 13. Sinuous Growth Path

in which $C_t = c_t^y L_t - c_t^o L_{t-1}$ is aggregate consumption in period t. Turn it into intensity form

$$\frac{K_{t+1}}{L_t} - \frac{K_t}{L_t} = F\left(\frac{K_t}{L_t}, 1\right) - \frac{C_t}{L_t} - \delta\frac{K_t}{L_t},$$

that is

$$k_{t+1}(1+n) = f(k_t) - c_t + (1-\delta)k_t. \tag{22}$$

However, what we care about for evaluation is *per capita* consumption $\frac{C_t}{L_t+L_{t-1}}$ instead of *per labor* consumption c_t (note that only the young generation works in each period) in the equation above. But with slight algebra

$$\frac{C_t}{L_t + L_{t-1}} = \frac{c_t L_t}{L_t + L_{t-1}} = \frac{c_t(1+n)L_{t-1}}{(1+n)L_{t-1} + L_{t-1}} = \frac{1+n}{2+n}c_t$$

we see that per capita consumption is just constant times of per labor consumption. So the maximization problem of per capita consumption is equivalent to the maximization problem of per labor consumption c_t.

In steady state $k_t = k_{t+1} = k^*$, from (22)

$$c^* = f(k^*) - (n+\delta)k^*.$$

The first order condition gives

$$\frac{dc^*}{dk^*} = f'(k^*) - (n + \delta) = 0,$$
$$k^*_{GOLDEN} = (f')^{-1} (n + \delta).$$

This implies that the steady state level of capital stock under golden rule only depends on two exogenous parameters n and δ. In contrast, what we got from the equilibrium of overlapping generation model is equation (15), implying that the steady state capital stock hinges on the agent's saving decision which may depend on the parameters more than n and δ.

To make it clear, we continue with our example. It's simple to compute the steady state level of capital stock under golden rule, given the parameters in the example

$$k^*_{GOLDEN} = (f')^{-1} (n + \delta) = \left(\frac{\alpha}{n + \delta}\right)^{\frac{1}{1-\alpha}}.$$

And we already got the equilibrium steady state capital stock

$$k^* = \left(\frac{\beta}{1 + \beta}\frac{1 - \alpha}{1 + n}\right)^{\frac{1}{1-\alpha}}.$$

Therefore, for any parameter set $(\beta, \alpha, n, \delta)$ such that

$$\frac{\alpha}{n + \delta} \neq \frac{\beta}{1 + \beta}\frac{1 - \alpha}{1 + n}$$

we would get $k^* \neq k^*_{GOLDEN}$, suggesting that the equilibrium allocation of overlapping generation model may not be Pareto optimal.

4.4.3 Dynamic Inefficiency

The discussion above simply shows us an example in which the competitive equilibrium allocation doesn't coincide with the Pareto optimal solution, in an economy with an infinite commodity space. This gives a chance in which the economic policy may improve the social welfare — Most of such discussions are left in our exercises.

5 Readings

Blanchard and Fischer (1989) Chapter 3.

6 Bibliographic Notes

The qualitative method for analyzing the economy's response to the policy change is sometimes sketched in the textbooks, but often not discussed in depth (For example one popular textbook states that "We ... focus on the simple case where the [policy] change is unexpected..."). Acemoglu (2009) is the only textbook which attempts to extend the Theorems of Welfare Economics to cope with the problems with infinite commodity spaces, and the classical form of the Theorems is well established in the mainstream advanced micro texts, e.g. Mas-Colell *et al*. (1995), Varian (1992), just to name a few.

The overlapping generation model is founded by Allais (1947), Samuelson (1958) and Diamond (1965). The section written here is just a summary of a wide selection of excellent chapters from Acemoglu (2009), Barro and Sala-i-Martin (2004), Blanchard and Fischer (1989) and Romer (2006). The model I presented is a much simplified but sufficiently generic version, and further interesting issues are to be explored in the exercises. Two recent works may be interesting: Galor and Ryder (1989) is an intensive study on the model's equilibrium properties, and de la Croix and Michel (2002) contribute a comprehensive framework for policy analysis based on overlapping generation models (note that the built-in imperfection of such models is the natural justification for economic policies).

7 Exercises

7.1 Ramsey Model

Consider the Ramsey model from Exercise 1.5 with $y = k^\alpha$ and $U_0 = \int_0^{+\infty} e^{-\rho t}\,(c(t))^\beta\,dt$, where $0 < \alpha < 1$ and $0 < \beta < 1$.

a)[A] Describe how each of the following *unexpected* changes affects the locus of $\dot{c} = 0$ and $\dot{k} = 0$ in a $k - c$ diagram, and how they affect the balanced-growth-path values of c and k:

i) A rise in β;

ii) A downward shift in the production function (a lower α);

iii) A rise in the rate of depreciation;

iv) A fall in the rate of time preference ρ.

b)[B] How do per-capita-consumption, capital intensity and the interest rate change during the adjustment processes if the initial capital intensity is at the initial steady state?

7.2 Ramsey Model for Decentralized Economy

Consider a Ramsey-Cass-Koopmans economy that is already on its balanced growth path. Suppose that the government introduces a tax on investment τ income at time $t = 0$, *unexpectedly*. Thus, the real interest rate that households face is now given by $r(t) = (1-\tau)f'(k(t))$. Assume that tax revenue is redistributed through lump sum transfers.

a)$^{\mathbf{A}}$ How does the tax affect the locus of $\dot{c} = 0$ and $\dot{k} = 0$?

b)$^{\mathbf{B}}$ How does the economy respond to the adoption of the tax at $t = 0$, and what are the dynamics afterwards?

c)$^{\mathbf{C}}$ Suppose there are many economies like this one, distinguished by different tax rates.

i) Show that the savings rate on the balanced growth path is decreasing in τ.

ii) Do citizens in high saving countries have an incentive to invest in low saving countries?

d)$^{\mathbf{B}}$ How, if at all, do the answers to part **a)** and **b)** change if the government does not rebate the tax revenue but instead uses it for government purchases?

Suppose that instead of announcing and implementing the tax at time $t = 0$, the government announces at $t = 0$ that it will begin to tax investment income at some later time t_1. Then

e)$^{\mathbf{A}}$ Draw the phase diagram showing the dynamics of c and k after time t_1.

f)$^{\mathbf{C}}$ Can c change discontinuously at t_1? Why or why not?

g)$^{\mathbf{B}}$ Draw the phase diagram showing the dynamics of c and k before t_1.

h)$^{\mathbf{B}}$ What must c do at time $t = 0$?

i)$^{\mathbf{B}}$ Summarize your results by sketching the paths of c and k as functions of time.

7.3 Ramsey Model with Technological Progress

Suppose that, in a Ramsey-Cass-Koopmans economy, production is given by the function $Y_t = F(K_t, e^{\phi t}L_t)$, where ϕ is the constant and exogenous rate of technical progress. Assume that the population grows at rate n and that the utility function is of constant relative risk aversion form, with a coefficient of relative risk aversion equal to γ.

a)$^{\mathbf{A}}$ Derive and interpret the modified golden rule condition in this case.

b)$^{\mathbf{A}}$ Characterize the dynamics of consumption and capital accumulation.

c)[B] Suppose that the economy is in steady state and that ϕ decreases *permanently* and *unexpectedly*. Describe the dynamic adjustment of the economy to this adverse supply shock.

7.4 Hand-to-Mouth Workers in the Ramsey Economy

Consider the following variantion of the simplest neoclassical growth model. Half of the population, the "hand-to-mouth"consumers, simply consume any labor income they earn each period — they never own any assets whatsoever. The other half, the "savers", have preferences and choices as in the standard neoclassical model. There is no population growth and we conveniently normalize the total population to be a continuum of size 2.

The preferences for the savers are standard

$$\sum_{t=0}^{+\infty} \beta^t u(c_t)$$

for some $\beta < 1$, and u twice continuously differentiable, increasing and strictly concave with the Inada condition $\lim_{c \to 0} u'(c) = +\infty$.

Technology is given by the constant returns to scale Cobb-Douglas production function

$$Y_t = K_t^{\alpha} L_t^{1-\alpha}.$$

Labor, L_t, is supplied inelastically by both types of agents each period with total labor supply normalized to 1. The savers and the hand-to-mouth agents each supply $\frac{1}{2}$.

The resource constraint is

$$C_t + K_{t+1} = Y_t + (1 - \delta) K_t$$

where $C_t = c_t^1 + c_t^2$ is aggregate consumption and c^1 represents consumption of hand to mouth consumers and c^2 consumption of savers. Note that we do not describe the preferences of the hand-to-mouth agents, just their behavior.

a)[A] Setup the standard description of markets for labor and capital, stating the budget constraints faced by savers and hand to mouth consumers, the (static) problem of the firm. Define a competitive equilibrium.

b)[B] Show that in equilibrium the labor income and consumption of the hand-to-mouth agents is a constant fraction λ of output Y_t. Determine λ.

c)[C] Argue that the competitive equilibrium is Pareto optimal for the "savers"in the following sense, it solves

$$\max_{\{c_t\}_{t=0}^{+\infty}} \sum_{t=0}^{+\infty} \beta^t u(c_t),$$
$$s.t.\ C_t + K_{t+1} = (1-\lambda) K_t^{\alpha} L_t^{1-\alpha} + (1-\delta) K_t$$

where λ is a constant fraction of output that goes to the hand-to-mouth agents found in **b)**.

d)$^{\mathbf{C}}$ Does the introduction of the hand-to-mouth consumers affect the steady state level of capital? Does the introduction of the hand-to-mouth consumers affect the equilibrium dynamics of consumption, output and capital relative to the case without hand-to-mouth consumers? Discuss: Stability, uniqueness of the steady-state, monotonicity and the speed of convergence to the steady state (Hint: For the speed of convergence take a linear approximation around the steady state with and without the hand-to-mouth consumers).

7.5 Labor-Leisure Choice in the Ramsey Economy

Consider a Ramsey growth model where the representative consumer maximizes

$$U = \int_0^{+\infty} e^{-\rho t} u\left[c(t), l(t)\right] dt, \rho > 0$$

with instantaneous preferences over consumption $c(t)$ and hours of work $l(t)$

$$u\left[c(t), l(t)\right] = \ln c(t) - \theta \frac{l(t)^{1+\eta} - 1}{1+\eta}, \theta > 0, \eta > 0.$$

Workers have a time endowment of one unit of labour, so $l(t) \in [0, 1]$ (Assume a constant population, normalised to 1.) Hours worked, $l(t)$, are combined with physical capital, $k(t)$, to produce output, $y(t)$, through a Cobb-Douglas technology

$$y(t) = k(t)^{\alpha} l(t)^{1-\alpha}, \alpha \in (0, 1).$$

Capital accumulation in the economy follows

$$\dot{k}(t) = i(t) - \delta k(t)$$

where $i(t)$ is investment and capital depreciates at rate δ. The goods market clears at all times t

$$y(t) = c(t) + i(t).$$

a)[A] State the planner's problem for this economy. Write down the Hamiltonian and derive the optimality conditions.

b)[A] Express consumption and the capital stock in terms of units of labour, and call the new normalized variables $\hat{c}(t)$ and $\hat{k}(t)$. Are the steady-state levels of consumption $\hat{c}(t)$ and capital $\hat{k}(t)$ different from the standard Ramsey model without endogenous labour supply? Explain.

c)[B] Derive the following expression for $\frac{\dot{l}}{l}$ as a function of $\frac{\dot{\hat{k}}}{\hat{k}}$ and $\frac{\dot{\hat{c}}}{\hat{c}}$

$$\frac{\dot{l}}{l} = -\frac{1}{1+\eta}\frac{\dot{\hat{c}}}{\hat{c}} + \frac{\alpha}{1+\eta}\frac{\dot{\hat{k}}}{\hat{k}}.$$

Provide some economic interpretion for this equation.

d)[C] By using the dynamic equations of $\frac{\dot{\hat{k}}}{\hat{k}}$ and $\frac{\dot{\hat{c}}}{\hat{c}}$ show that

$$\frac{\dot{l}}{l} = \frac{\alpha}{\alpha+\eta}\left[\frac{\hat{c}^*}{\hat{k}^*} - \frac{\hat{c}(t)}{\hat{k}(t)}\right].$$

Show that when $\hat{k}(0) < \hat{k}^*$, $\frac{\hat{c}(t)}{\hat{k}(t)}$ declines monotonically along the saddle path to the steady state. Explain whether the speed of convergence of $\hat{k}(t)$ to the steady state is larger or smaller than for the standard Ramsey model.

7.6 Two-Sector Neoclassical Growth Model

Consider an economy with a representative consumer whose preference is described by

$$\sum_{t=0}^{+\infty} \beta^t \frac{c_t^{1-\sigma}}{1-\sigma}.$$

a)[C] (Two-Sector neoclassical growth model) Assume that there are separate technologies for producing the consumption good and the investment good. Capital and labor can be moved freely across the two sectors. The resource constraints are given by

$$\begin{aligned} c_t &\leq l_c^{\alpha} k_c^{1-\alpha}, \\ i_t &\leq l_i^{\gamma} k_i^{1-\gamma}, \\ l_{it} + l_{ct} &\leq 1, \\ k_{it} + k_{ct} &\leq k_t, \\ k_{t+1} &= (1-\delta)k_t + i_t. \end{aligned}$$

Write down the planning problem for this economy. Derive the first order conditions and characterize the steady state.

b)[C] (Two-Sector neoclassical growth model with immobile capital) Now consider the same model under the additional assumption that there are two different types of capital, one used in the consumption goods sector and one in the investment goods sector. The two types of capital cannot be transformed into each other. The resource constraints are given by

$$
\begin{aligned}
c_t &\leq l_c^{\alpha} k_c^{1-\alpha}, \\
i_{it} + i_{ct} &\leq l_i^{\gamma} k_i^{1-\gamma}, \\
l_{it} + l_{ct} &\leq 1, \\
k_{it+1} &= (1-\delta)k_{it} + i_{it}, \\
k_{ct+1} &= (1-\delta)k_{ct} + i_{ct}.
\end{aligned}
$$

Write down the planning problem for this economy. Derive the first order conditions and characterize the steady state.

7.7 *Overlapping Generations: Properties of Steady States, Poverty Trap*

Consider an overlapping generation economy with the agents' generational preferences featured by

$$u_t = u\left(c_t^y, c_{t+1}^o\right) = \left(c_t^y c_{t+1}^o\right)^{\frac{1}{2}} .$$

The production function is defined as

$$
f(k) = \begin{cases} 2 + 5k - k \ln k & \text{for } 0 < k \leq 4 \\ -10 + (8 + 3\ln 4)k - 4k \ln k & \text{for } 4 < k \leq 6 \\ 8 + (5 + 3\ln 4 - 3\ln 6)k - k \ln k & \text{for } 6 < k \leq 10. \end{cases}
$$

Normalize the population size to be one and assume a population growth rate of zero.

a)[A] Is $f(k)$ a neoclassical production function?

b)[A] Derive the first order conditions for the profit maximizing production sector. Show that these conditions give rise to a continuous, positive, and decreasing function $r(k)$ and to a continuous, positive and increasing function $w(k)$.

c)[B] Calculate the optimal solutions for c_t^y and c_{t+1}^o. Provide some intuitions for your results.

d)[C] Characterize the dynamics of k_t, i.e. specify the function $k_{t+1} = \phi(k_t)$. Find the steady state(s) of the economy, and interpret your findings.

7.8 Overlapping Generations: Dynamic Inefficiency

Consider the following economy of overlapping generations of vegetarians. Assume there is an equal number of young and old individuals and population N is constant. Each individual is endowed with one vegetable at birth. While young, the individual decides how much of the vegetable to eat, and how much to plant (there are no refrigerators so it can not be stored). If a fraction s of the vegetable is planted, then As^{α} vegetables will grow for second-period consumption, in which $A > 0$ and $0 < \alpha < 1$. The portion planted cannot be eaten (this is equivalent to 100% depreciation). Utility is of the form:

$$u(c_{1t}, c_{2t+1}) = \ln c_{1t} + \ln c_{2t+1}.$$

a)[A] Write down the individual's decision problem. How much does the individual consume and save in the first period?

b)[A] Can trade between generations take place in this economy? Describe the equilibrium in this economy.

c)[A] Suppose that in each period the government confiscates a fraction f of each young person's vegetable, and distributes the proceeds equally among the old individuals. Find expressions for first and second period consumption, c_{1t} and c_{2t+1}, in this case.

d)[C] Conider a small increase in f starting from $f = 0$. Derive an expression for welfare as a result of this policy.

e)[B] Under what conditions is the policy in part **d)** welfare improving? Provide a brief intuitive explanation.

7.9 Social Security and Capital Accumulation in Overlapping Generations Model

Consider the basic overlapping generations model with two-period lived individuals as described in our class notes. To simplify assume a time-separable utility function

$$u(t) = u\left(c_t^y\right) + \beta u\left(c_{t+1}^o\right), \beta \in (0, 1).$$

The economy has a social security system. Denote d_t the contribution of a young person at t and b_t the benefit received by an old person at t. Thus, the budget constraints faced by each generation t become

$$c_t^y + s_t \leq w_t - d_t,$$
$$c_{t+1}^o \leq (1 + r_{t+1})\, s_t + b_{t+1}.$$

We distinguish between two types of social security systems: one is *fully funded*, the other is *unfunded* (the latter is often called *pay-as-you-go* system).

The fully funded system invests the contributions of the young at t and returns the investment with interest at $t + 1$ to the then old. Accordingly, it holds that

$$b_{t+1} = (1 + r_{t+1})\, d_t.$$

The pay-as-you-go system transfers current contributions made by the young to the current old so that

$$b_t = \frac{L_t}{L_{t+1}} d_t. \tag{23}$$

a)[A] Suppose the economy is in a steady state with $k_t = k^*$.

(1) Determine and compare the rate of return on per capita contributions that each generation can expect under the fully funded system and under the pay-as-you-go system. Interpret your findings.
(2) How does your result change if we add exogenous labor-saving technical change, i. e., assume a final-good production function $Y_t = F(K_t, A_t L_t)$ where $A_t = (1+x)A_{t-1}$, $A_0 > 0$ and $x > -1$ given. Consider a steady state $\hat{k}_t = \frac{k_t}{A_t} = \hat{k}^*$ and a contribution rate $0 < \tau < 1$ such that $d_t = \tau w_t$.

b)[A] Consider an economy with a fully funded system. Show the following result: if $d_t \leq (1 + n)k_{t+1}$ for all t, then the fully funded social security system has no effect on total savings and capital accumulation. In other words, the equilibrium path of the economy is independent of whether it runs a fully funded social security system or none. Interpret this result.

c)[C] Consider an economy with a pay-as-you-go system.

(1) Set up the Euler equation of a representative individual of generation t.
   (a) Study the comparative-static effect (keeping w_t and r_{t+1} constant) of the obligation to pay contributions when young, $d_t > 0$, and the expectation to receive transfers when old, $(1 + n)d_{t+1} > 0$ (here use (23)), on the incentives to save.
   (b) Suppose $d_t = d_{t+1} = d$. Show that

$$\left|\frac{ds_t}{dd}\right| \gtrless 1 \Leftrightarrow n \gtrless r_{t+1}.$$

(2) Argue that the evolution of k_t can be stated as

$$(1 + n)k_{t+1} = s(w(k_t), r(k_{t+1}), d). \tag{24}$$

(3) Suppose (24) defines a well-behaved relationship between k_t and k_{t+1} (such that the steady state exists). Show that

$$\frac{dk_{t+1}}{dd_t} < 0.$$

Discuss the implications of this result for capital accumulation. Is this a desirable outcome?

7.10 Overlapping Generations with Money

(Samuelson, 1958) Suppose, as in the Diamond (1965) model, that N_t 2-period-lived individuals are born in period t and that generations are growing with rate n. The utility function of a representative individual is $U_t = \ln c_{1,t} + \ln c_{2,t+1}$. Each individual is born with an endowment of A units of the economy's single good. The good can either be consumed or stored. Each unit stored yields $x > 0$ units next period.

In period 0, there are N_0 young individuals and $\frac{1}{1+n}N_0$ old individuals endowed with some amount Z of the good. Their utility is simply $c_{2,0}$.

a)[A] Describe the decentralized equilibrium of this economy. (Hint: Will members of any generation trade with members of another generation?)

b)[A] Consider paths where the fraction of agents endowment that is stored, s_t, is constant over time. What is per capita consumption (weighted average from young and old) on such a path as a function of s?

c)[B] If $x < 1 + n$, which value of $s \in [0, 1]$ is maximizing per capita consumption?

d)[B] Is the decentralized equilibrium Pareto-efficient? If not, how could a social planner raise welfare?

Suppose now that old individuals in period 0 are also endowed with M units of a storable, divisible commodity, which we call money. Money is not a source of utility. Assume $x < 1+n$.

e)[C] Suppose the price of the good in units of money in periods t and $t + 1$ is given by P_t and P_{t+1}, respectively. Derive the demand functions of an individual born in t.

f)[C] Describe the set of equilibria.

g)[C] Explain, why there is an equilibrium with $P_t \to +\infty$. Explain why this must be the case if the economy ends at some date T that is common knowledge among all generations.

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