3

Dynamic Systems

For want of a nail the shoe is lost
For want of a shoe the horse is lost
For want of a horse the rider is lost
For want of a rider the battle is lost
For want of a battle the kingdom is lost
And all for the loss of a horseshoe nail

—George Herbert (1633), *Jacula Prudentum*

1 Introduction

The dynamic of an economic system often results in a system of differential equations as following:

$$\dot{\mathbf{x}}(t) = \mathbf{F}(\mathbf{x}(t)),$$

which is called a *system of first-order autonomous ordinary differential equations*. This is a simply concise form of

$$\begin{cases} \dot{x}_1(t) = f_1(x_1(t), \ldots, x_n(t)), \\ \dot{x}_2(t) = f_2(x_1(t), \ldots, x_n(t)), \\ \cdots\cdots \\ \dot{x}_n(t) = f_n(x_1(t), \ldots, x_n(t)). \end{cases}$$

The dynamic of any variable $x_i(t)(\forall i \in \{1, \ldots, n\})$ is jointly determined by a function $f_i(x_1(t), \ldots, x_n(t)) : \mathbb{R}^n \to \mathbb{R}$ of all variables in this system. The system is *autonomous* because f_i doesn't involve any variable other than $x_1(t), \ldots, x_n(t)$. The differential equations are first-order with respect to time, for normally in economics we only study the *speed* of the variables' adjustment rather than *acceleration*.

The value $\mathbf{x}^*$ such that

$$\mathbf{F}(\mathbf{x}^*(t)) = 0 \tag{1}$$

is called *steady-state value* of $\mathbf{x}$ since $\mathbf{x}(t)$ doesn't change overtime. The concept of steady-state is of special interest in macroeconomics as equilibrium concept, for one may think of it as a state at which an economy ends given sufficiently long time. And since such a state is time-invariant, people often take it as a baseline for policy analysis.

If we think about the concept over, it's natural to raise two more direct concerns:

- Existence. One may ask whether such a steady state exists at all, i.e. whether there is an equilibrium at which the economy stays with time-invariant properties. Usually this is not difficult to see by verifying the existence of solution for equation (1);
- Reachability, or stability. Suppose that steady state exists. Then one may ask, given the initial condition $\mathbf{x}(0) = (x_1(0), \ldots, x_n(0))$, whether there is a *path* $G(\mathbf{x}(t)) = 0$ by which the economy can reach the steady state (problem of reachability). For example, given the initial conditions $\mathbf{x}(0)$ (which may contain today's level of consumption, capital stock, labor supply etc.) of a developing country we can ask whether people can choose the proper

values of $\mathbf{x}(t)$ from now on (i.e. the path) and reach the state $\mathbf{x}^*$ they desire (i.e. steady state). A similar (but not the same) question might be whether the system stays around the steady state or diverge if we perturb a system in the steady state (problem of stability – think about the horseshoe nail). These two questions are closely related in mathematics, and are the central issue in the following lecture.

In the following sections we provide two approaches to analyzing a system of first-order autonomous ordinary differential equations. The first is a graphical analysis using the *phase diagram*. The advantage of this approach is that it is simple, economically founded and provides a qualitative, sometimes semi-quantitive solution. Furthermore, this approach works for both nonlinear and linear systems. The main drawback is that it works best for two-dimensional systems (occasionally one may try to work it out for three-dimensional systems, but usually it's not suitable for even higher dimensions). The other is an analytical approach. The advantages of it are that it gives quantitative results and can be applied in larger systems. The disadvantage is that it works best for linear equations, and one needs much more effort in treating nonlinear systems with analytical approach.

As a least technical exposure, in this chapter we only talk about two-dimensional systems which should be sufficient for most applications in this course. The facts for systems of higher dimensions are given in the appendix.

2 Graphical Analysis

For a system of autonomous ordinary differential equations as the one we concern

$$\begin{cases} \dot{x}_1(t) = f_1(x_1(t), \dots, x_n(t)), \\ \dot{x}_2(t) = f_2(x_1(t), \dots, x_n(t)), \\ \dots\dots \\ \dot{x}_n(t) = f_n(x_1(t), \dots, x_n(t)), \end{cases}$$

the space without t dimension, $(x_1, \dots, x_n) \in \mathbb{R}^n$, is called *phase space*. The basic idea behind graphical methods is to identify the driving forces from each $x_i, \forall i \in \{1, \dots, n\}$ by plotting the *phase diagram* in the phase space. We show this procedure by an example, and readers are asked to do more exercises in Problem Set 1 and 2.

The following result is what we got from a Ramsey-Cass-Koopmans model in which the production function is neo-classical and there is neither population growth nor technological progress.

$$\begin{cases} \dot{k}(t) = f(k(t)) - c(t) - \delta k(t), \\ \frac{\dot{c}(t)}{c(t)} = f'(k(t)) - \delta - \rho. \end{cases}$$

$(k, c) \in \mathbb{R}^2_+$ forms a two-dimensional phase space. The steady state (k^*, c^*) is the point where $\dot{k}(t) = 0$ as well as $\dot{c}(t) = 0$, i.e. there is no driving force, from either k or c, pushing the system to elsewhere when the system stands exactly at (k^*, c^*). Then the problem of stability is equivalent to:

$\forall (k, c) \in \mathbb{R}^2_+$ and $(k, c) \neq (k^*, c^*)$, where will the system go with driving forces from k and c?

If we can identify such driving forces for every point in the phase space, we are able to say something about stability. For example the system is surely stable if it is driven towards (k^*, c^*) at any (k, c).

2.1 The Dynamics of k

First let's have a look on the driving forces from k, given $\dot{k}(t) = f(k(t)) - c(t) - \delta k(t)$. Let's begin with finding the points on which k doesn't play a role, i.e. $\dot{k}(t) = 0$. Then we can write c as a function of k: $c(t) = f(k(t)) - \delta k(t)$, meaning that c is the difference between $f(k)$ and δk as the blue-shaded part in Figure 1, plotted as $\dot{k}(t) = 0$ curve in Figure 2. For any point on $\dot{k}(t) = 0$ curve (e.g. point 1 in Figure 2) k doesn't change over time.

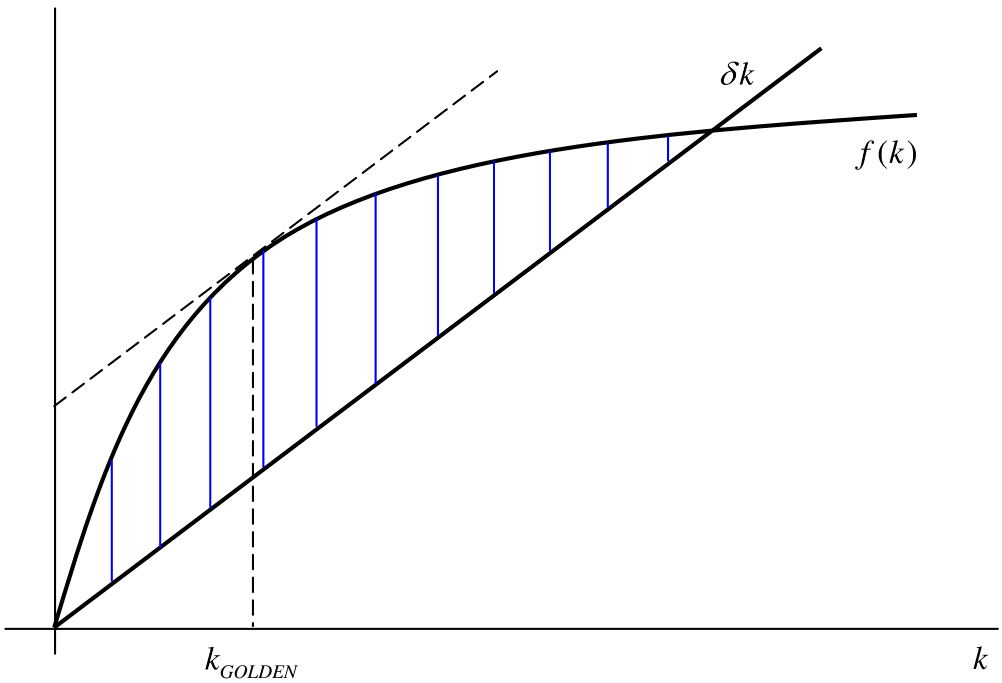

Fig. 1. $k - c$ Relation under $\dot{k} = 0$

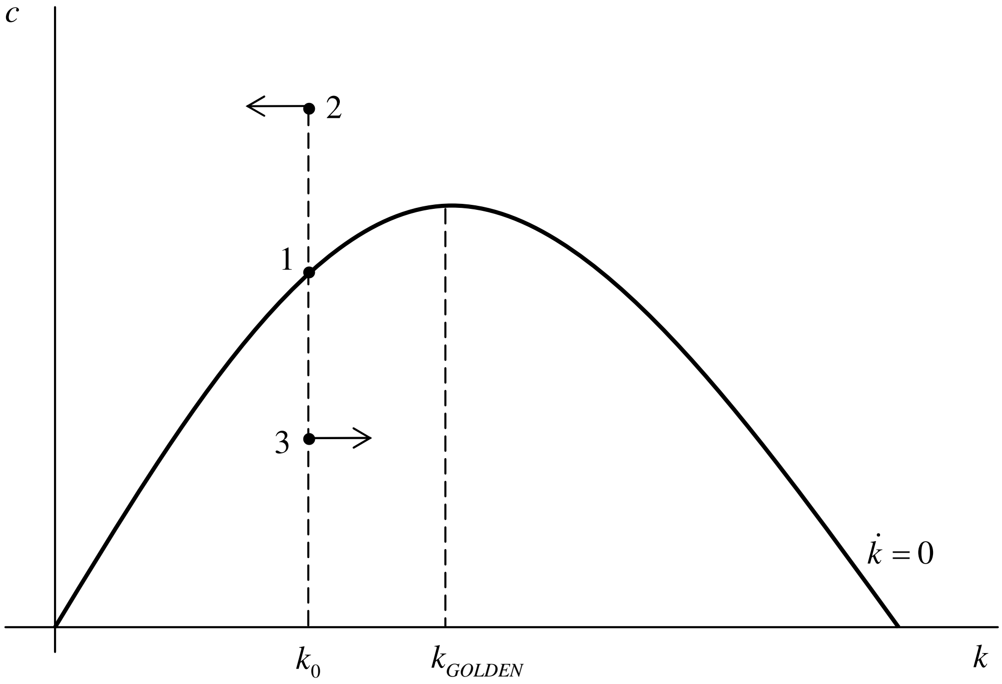

Fig. 2. The Dynamics of k

Then let's take an arbitrary point above this curve (e.g. point 2 in Figure 2), i.e. c is larger than $f(k) - \delta k$. From $\dot{k}(t) = f(k(t)) - c(t) - \delta k(t)$ larger c makes $\dot{k}(t) < 0$ meaning that k decreases overtime. Therefore for any point above $\dot{k}(t) = 0$ there is a driving force from k pushing the system to the direction with lower k. The similar argument can be stated for the point below $\dot{k}(t) = 0$ (e.g. point 3 in Figure 2).

2.2 *The Dynamics of c*

Any point on which c doesn't play a role lies on the curve $\dot{c}(t) = 0$, which simply mean that $f'(k(t)) - \delta - \rho = 0$. Since $f(k)$ is strictly monotone, k can be solved as $k^* = (f')^{-1}(\delta + \rho)$ as plotted in Figure 3.

For an arbitrary point to the left this curve k becomes less, then $\frac{\dot{c}(t)}{c(t)} = f'(k(t)) - \delta - \rho > 0$ because of $f'(k) > 0$ and $f''(k) < 0$. c increases over time as point 2 in Figure 3. Similarly c decreases for the points to the right of $\dot{c}(t) = 0$, such as point 3 in Figure 3.

2.3 *The Phase Diagram*

From the analysis above we know that for each point (c, k) the system is driven by the forces from c and k. Now we combine Figure 2 and Figure 3 to see how the system behaves under

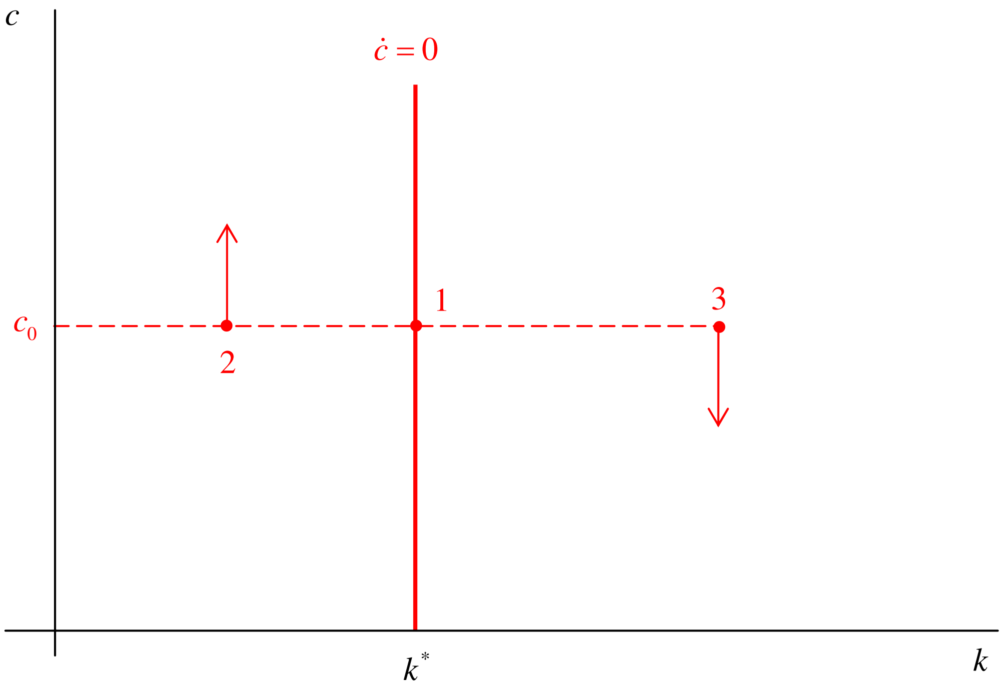

Fig. 3. THE DYNAMICS OF c

the aggregate force, as shown in FIGURE 4. $\dot{k}(t) = 0$ and $\dot{c}(t) = 0$ split the phase space into four parts, A, B, C and D. The joint of these two curves, E, is simply the steady state (the readers will be asked why $k^* < k_{GOLDEN}$ in the PROBLEM SET). Take an arbitrary point (1) from A as an example. Since it is located above $\dot{k}(t) = 0$, it is pushed to the west by k. And since it is also located to the left of $\dot{c}(t) = 0$, it is pushed to the north by c. As a result, the system starting from this point would be pushed to north-west, i.e. getting lower in k and higher in c. The readers can try to analyze the behaviour of other representative points, from (2) to (8).

If one plots the forces for more points, as shown in FIGURE 5, one (or one's computer) can find the main characteristics of the system:

- The system is unstable, because there are forces driving the system away from the steady state (especially in regions A and D);
- There are two trajectories going through the steady state. One leads the system towards the steady state, called *stable arm*; the other leads the system away from the steady state, called *unstable arm*.

These facts are very interesting – although the system is basically unstable, we are still able to approach the steady state if we manage to set the right (k, c) following the stable arm. Such steady state is called a *saddle path equilibrium* since a system on the *saddle path* is extremely sensitive to perturbations. For example, in FIGURE 6 starting from any $k(0) < k^*$ the only possible way to approach E is to choose $c(0)$ defined by the saddle path. Any higher (as (1)) or lower (as (3)) c leads the system to inferior solutions. The same argument also holds for those $k'(0) > k^*$.

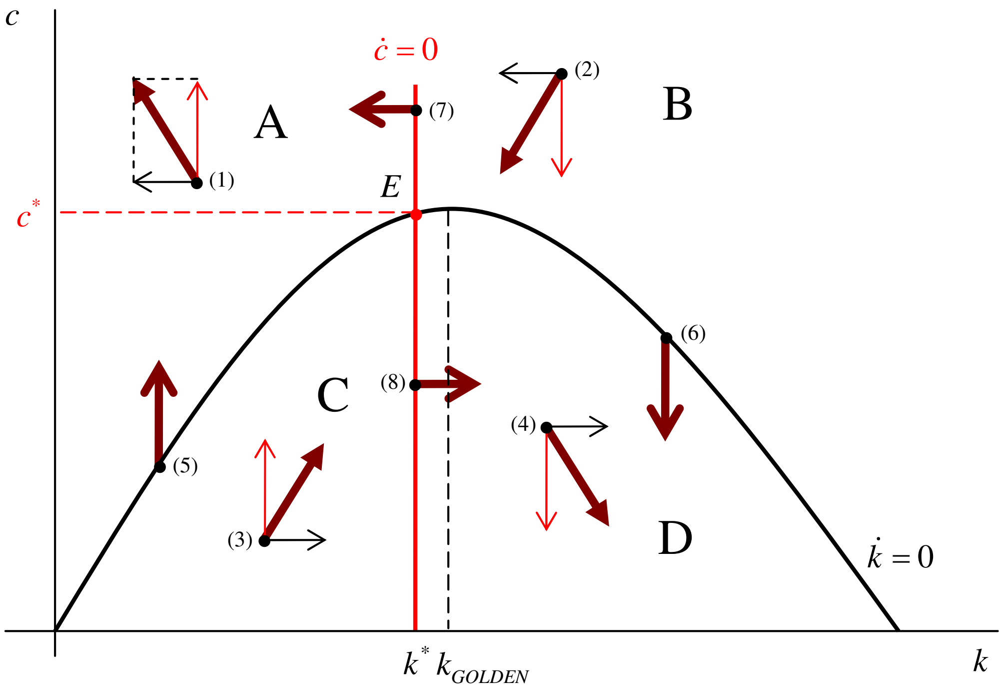

Fig. 4. The Dynamics of c and k

3 Analytical Analysis

We can also analyze the stability problem in an analytical way. The simplest and most widely studied case is the system of linear differential equations. Since it gives straight-forward quantitative results, in practice people tend to set up a system in a linear way or transform a non-linear system into a linear one. In this section we will expose a complete characterization for two-dimensional linear systems, and you will see the same patterns of stability as those we explored in the previous section. Then we use log-linearization techniques upon non-linear systems, which allow us to apply our knowledge of linear systems for non-linear cases.

3.1 Linear Systems

Consider a general two-dimensional linear system

$$\begin{cases} \dot{x}_1(t) = a_{11}x_1(t) + a_{12}x_2(t), \\ \dot{x}_2(t) = a_{21}x_1(t) + a_{22}x_2(t), \end{cases}$$

in which $a_{ij} \in \mathbb{R}$ $(i, j \in \{1, 2\})$ as well as

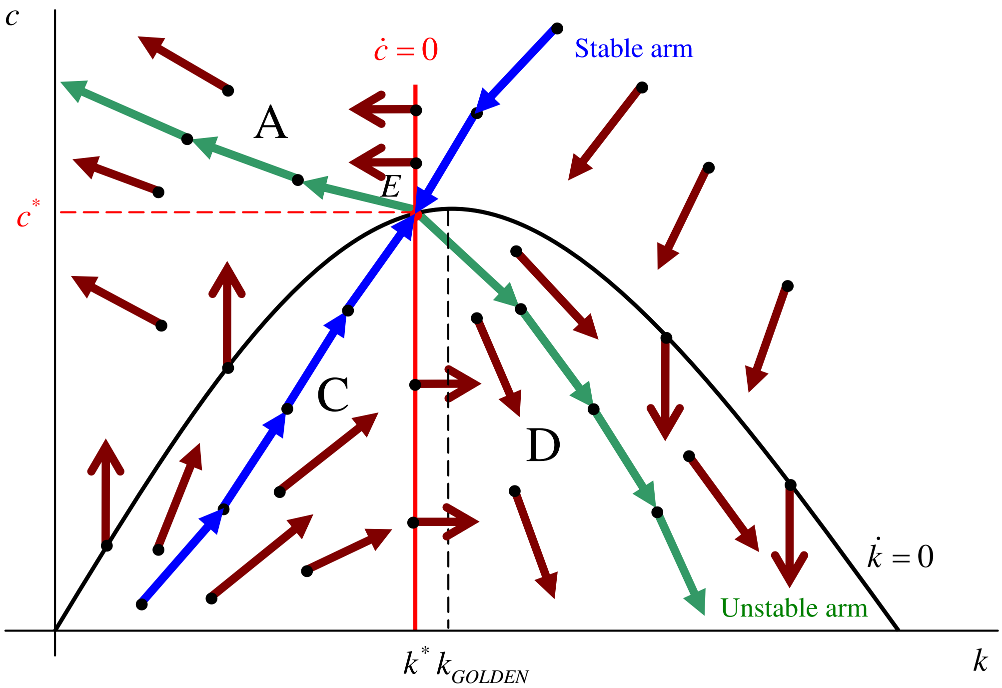

Fig. 5. The Field of Forces

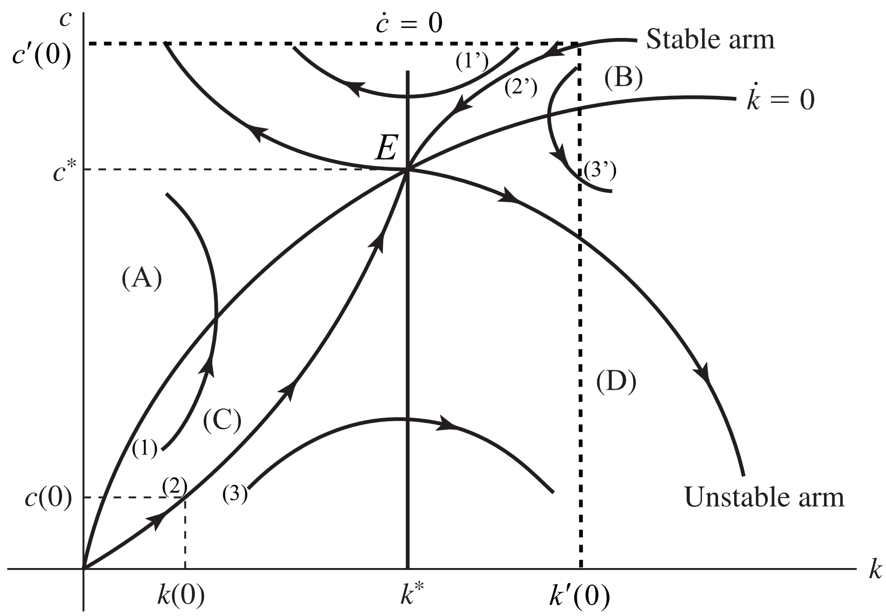

Fig. 6. The Phase Diagram with a Saddle Path

$$\det \mathbf{A} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \neq 0.$$

$\mathbf{A}$ is defined as the coefficient matrix

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}.$$

Compute eigenvalues λ_1 and λ_2 from characteristic equation

$$\begin{vmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{vmatrix} = 0,$$

and this is equivalent to solving a quadratic equation in λ

$$\begin{aligned} (a_{11} - \lambda)(a_{22} - \lambda) - a_{12}a_{21} &= 0, \\ \lambda^2 - (a_{11} + a_{22})\lambda + a_{11}a_{22} - a_{12}a_{21} &= 0, \\ \lambda^2 - \mathrm{tr}\mathbf{A}\lambda + \det \mathbf{A} &= 0. \end{aligned}$$

By discriminant $\Delta = \mathrm{tr}^2\mathbf{A} - 4\det \mathbf{A}$ one can see that

- If $\Delta \geq 0$ the equation has two real roots

$$\lambda_{1,2} = \frac{\mathrm{tr}\mathbf{A} \pm \sqrt{\Delta}}{2};$$

- If $\Delta < 0$ the equation has two complex roots

$$\lambda_{1,2} = \frac{\mathrm{tr}\mathbf{A} \pm \sqrt{|\Delta|}i}{2},$$

and $\lambda_1 + \lambda_2 = \mathrm{tr}\mathbf{A}$, $\lambda_1\lambda_2 = \det \mathbf{A}$.

Then apply λ_1 and λ_2 respectively into the following equations

$$\begin{bmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = 0$$

and solve $(u_1, u_2) = (\alpha_1, \alpha_2)$ for λ_1 as well as $(u_1, u_2) = (\beta_1, \beta_2)$ for λ_2.

(1) The eigenvalues are real numbers, i.e. $\Delta \geq 0$ $\lambda_1, \lambda_2 \in \mathbb{R}$. The general solution takes the form as following

$$\begin{cases} x_1 = c_1\alpha_1 \exp(\lambda_1 t) + c_2\beta_1 \exp(\lambda_2 t), \\ x_2 = c_1\alpha_2 \exp(\lambda_1 t) + c_2\beta_2 \exp(\lambda_2 t), \end{cases} \tag{2}$$

in which c_1 and c_2 are constants.

- If $\lambda_1 < 0, \lambda_2 < 0$ i.e. $\lambda_1 + \lambda_2 = \mathrm{tr}\mathbf{A} < 0$, $\lambda_1\lambda_2 = \det \mathbf{A} > 0$, the system is asymptotically stable as Figure 7 shows. The steady state $\left(x_1^*, x_2^*\right)$ is called *stable node*;

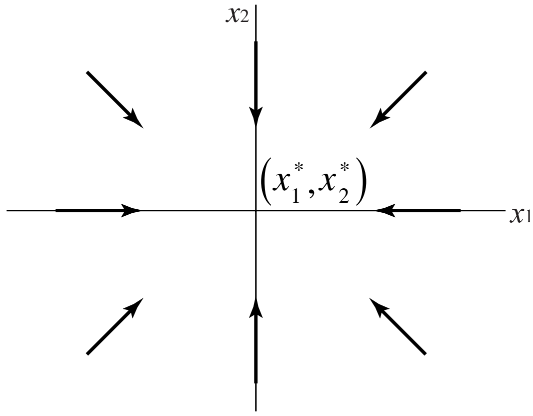

Fig. 7. Stable System: $\lambda_1 < 0, \lambda_2 < 0$

- If $\lambda_1 > 0, \lambda_2 > 0$ i.e. $\lambda_1 + \lambda_2 = \mathrm{tr}\mathbf{A} > 0$, $\lambda_1\lambda_2 = \det \mathbf{A} > 0$, the system is unstable as Figure 8 shows. The steady state $\left(x_1^*, x_2^*\right)$ is called *unstable node*;

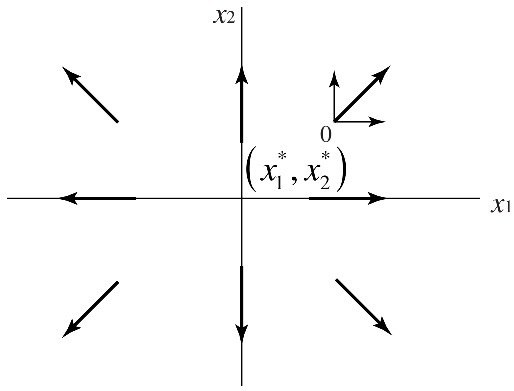

Fig. 8. Unstable System: $\lambda_1 > 0, \lambda_2 > 0$

- If $\lambda_1 > 0, \lambda_2 < 0$ i.e. $\lambda_1\lambda_2 = \det \mathbf{A} < 0$, the system is unstable as Figure 9 shows. However there exists a *saddle path* leading to the steady state. Such steady state is called *saddle point*.

(2) ★ The eigenvalues are complex numbers, i.e. $\Delta < 0$, $\lambda_1, \lambda_2 \notin \mathbb{R}$ and $\lambda_1 = p + iq$, $\lambda_2 = p - iq$ with $q \neq 0$. The general solution takes the form as following

$$\begin{cases} x_1 = \exp(pt)(c_1 \cos qt + c_2 \sin qt), \\ x_2 = \exp(pt)(c_1' \cos qt + c_2' \sin qt), \end{cases}$$

in which c_1 and c_2 are constants, and c_1' and c_2' are their linear combinations.

- If $\lambda_1 = p + iq$, $\lambda_2 = p - iq$ with $p < 0$ and $q \neq 0$, i.e. $\lambda_1 + \lambda_2 = \mathrm{tr}\mathbf{A} < 0$, the system is oscillatingly stable as Figure 10 shows. The steady state $\left(x_1^*, x_2^*\right)$ is called *stable spiral*

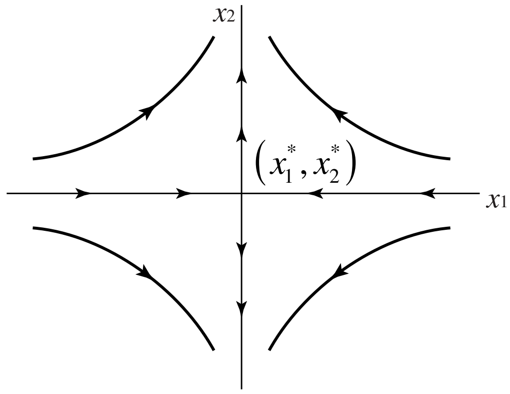

Fig. 9. Unstable System with Saddle Path: $\lambda_1 > 0, \lambda_2 < 0$

point;

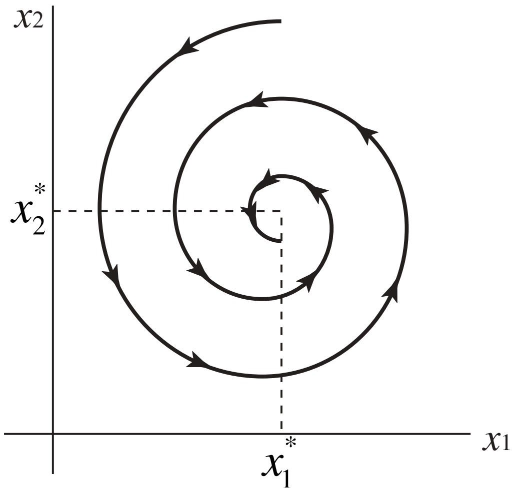

Fig. 10. Stable Oscillating System: $\text{Re}\lambda_1 < 0, \text{Re}\lambda_2 < 0$

- If $\lambda_1 = p + iq$, $\lambda_2 = p - iq$ with $p > 0$ and $q \neq 0$, i.e. $\lambda_1 + \lambda_2 = \text{tr}\mathbf{A} > 0$, the system is oscillatingly unstable as Figure 11 shows. The steady state $\left(x_1^*, x_2^*\right)$ is called *unstable spiral point*;
- If $\lambda_1 = p + iq$, $\lambda_2 = p - iq$ with $p = 0$ and $q \neq 0$, i.e. $\lambda_1 + \lambda_2 = \text{tr}\mathbf{A} = 0$, the system is oscillatingly stable as Figure 12 shows. The steady state $\left(x_1^*, x_2^*\right)$ is called *center*.

Figure 13 is a summary of all the cases.

3.2 Non-Linear Systems

The analysis of a non-linear system needs more rigrous maths. But in economics we often concern more about the system's behaviour around the steady state. So people log-linearize the system around the steady state by using first-order Taylor expansion to get an linear sys-

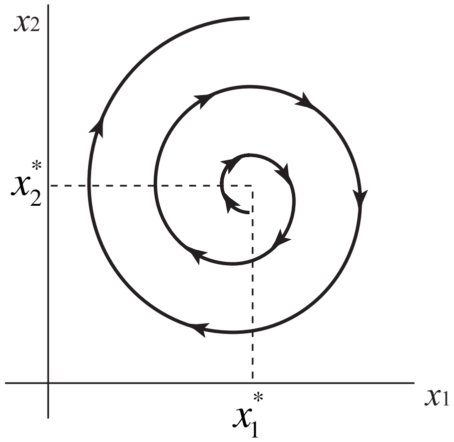

Fig. 11. Unstable Oscillating System: $\text{Re}\lambda_1 > 0$, $\text{Re}\lambda_2 > 0$

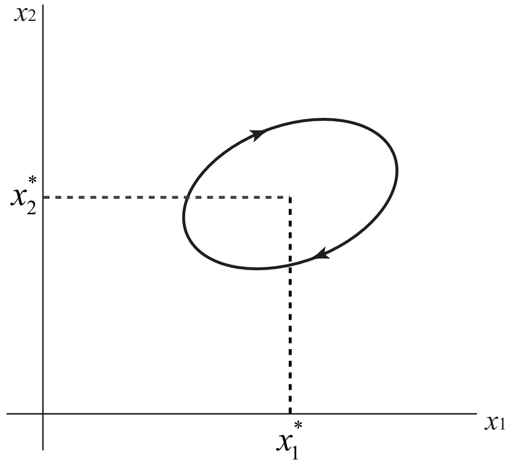

Fig. 12. Oscillating System: $\text{Re}\lambda_1 = 0$, $\text{Re}\lambda_2 = 0$

tem as an approximation. Then one can apply all the lessons we have learned in the previous section.

3.2.1 Log-Linearization

Log-linearization is a very useful trick in macroeconomic analysis by using the nice properties of logarithm functions. Suppose that X_t is a strictly positive variable, and X is its steady state. Define

$$x_t = \ln X_t - \ln X$$

as the logarithmic deviation. Notice that $\lim_{\kappa \to 0} \ln(1 + \kappa) = \kappa$, then

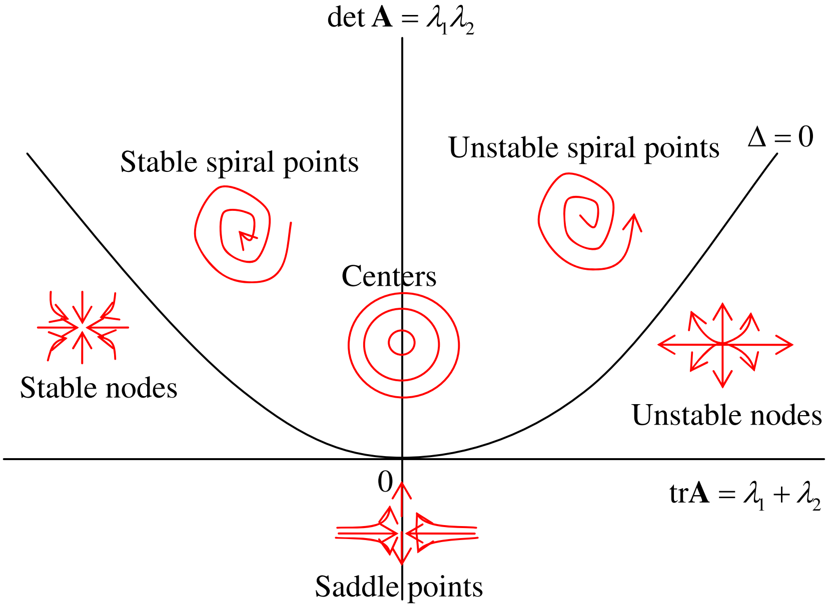

Fig. 13. Summary

$$x_t = \ln\left(\frac{X_t}{X}\right) = \ln\left(1 + \frac{X_t - X}{X}\right) \approx \frac{X_t - X}{X},$$

i.e. x_t is approximately the percentage deviation of X_t from the steady state when such deviation is small, capturing the *local* behavior around the steady state.

Now let's consider a general functional form. Suppose that we have an equation as following

$$f(A_t, B_t, \ldots) = g(Z_t)$$

in which $A_t, B_t, \ldots, Z_t$ are strictly positive variables, and $f(\cdot)$, $g(\cdot)$ may be non-linear in the variables. Also this equation has a steady state such that

$$f(A, B, \ldots) = g(Z).$$

To implement log-linearization around steady state rewrite the equation using the fact that $X_t = \exp(\ln X_t)$ and then take logs on both sides

$$\ln f(\exp(\ln A_t), \exp(\ln B_t), \ldots) = \ln g(\exp(\ln Z_t)).$$

Take the first order Taylor approximation around the steady state $(\ln(A), \ln(B), \ldots, \ln(Z))$

$$\ln f(A, B, \ldots) + \frac{1}{f(A, B, \ldots)}\left[\frac{\partial f(A, B, \ldots)}{\partial A} A(\ln A_t - \ln A) + \frac{\partial f(A, B, \ldots)}{\partial B} B(\ln B_t - \ln B) + \ldots\right]$$

$$= \ln g(Z) + \frac{1}{g(Z)} \left[g'(Z) Z (\ln Z_t - \ln Z) \right],$$

using the definition $x_t = \ln X_t - \ln X$ and rearrange the equation above one can get

$$\frac{\partial f(A, B, \ldots)}{\partial A} A a_t + \frac{\partial f(A, B, \ldots)}{\partial B} B b_t + \ldots = g'(Z) Z z_t,$$

Since $A, B, \ldots, Z$ are constants, the original non-linear equation is adapted to a new equation that is linear in $a_t, b_t, \ldots, z_t$.

3.2.2 Example

Consider the example exposed in Section 2. For simplicity assume that the production function is Cobb-Douglas.

$$\begin{cases} \frac{\dot{c}(t)}{c(t)} = \alpha k(t)^{\alpha-1} - \delta - \rho, \\ \dot{k}(t) = k(t)^{\alpha} - c(t) - \delta k(t). \end{cases}$$

Rewrite the expression of $\frac{\dot{c}(t)}{c(t)}$ into log-linearized form

$$\frac{d \ln c}{dt} = \alpha \exp\left[(\alpha - 1) \ln k\right] - \delta - \rho, \tag{3}$$

as well as the expression of $\frac{\dot{k}(t)}{k(t)}$

$$\frac{d \ln k}{dt} = \exp\left[(\alpha - 1) \ln k\right] - \exp\left[\ln\left(\frac{c}{k}\right)\right] - \delta. \tag{4}$$

Applying first order Taylor expansion to these two equations around steady state (c^*, k^*), it's simple to get $\frac{\dot{c}}{c}$ from (3)

$$\begin{aligned} \frac{d \ln c}{dt} &= \alpha(\alpha - 1) \exp\left[(\alpha - 1) \ln k^*\right] \left[\ln\left(\frac{k}{k^*}\right)\right] \\ &= (\alpha - 1)(\rho + \delta) \left[\ln\left(\frac{k}{k^*}\right)\right], \end{aligned}$$

as well as $\frac{\dot{k}}{k}$ from (4)

$$\frac{d\ln k}{dt} = \left\{(\alpha-1)\exp\left[(\alpha-1)\ln k^*\right] + \exp\left[\ln\left(\frac{c^*}{k^*}\right)\right]\right\}\left[\ln\left(\frac{k}{k^*}\right)\right]$$
$$- \exp\left[\ln\left(\frac{c^*}{k^*}\right)\right]\left[\ln\left(\frac{c}{c^*}\right)\right]$$
$$= \rho\left[\ln\left(\frac{k}{k^*}\right)\right] - \left(\frac{\rho+\delta}{\alpha} - \delta\right)\left[\ln\left(\frac{c}{c^*}\right)\right].$$

For simplicity, rewrite the linearized system as

$$\begin{bmatrix} \frac{d\ln k}{dt} \\ \frac{d\ln c}{dt} \end{bmatrix} = \begin{bmatrix} \rho & -\left(\frac{\rho+\delta}{\alpha} - \delta\right) \\ (\alpha-1)(\rho+\delta) & 0 \end{bmatrix} \begin{bmatrix} \ln\left(\frac{k}{k^*}\right) \\ \ln\left(\frac{c}{c^*}\right) \end{bmatrix} = \mathbf{A} \begin{bmatrix} \ln\left(\frac{k}{k^*}\right) \\ \ln\left(\frac{c}{c^*}\right) \end{bmatrix}.$$

To find the eigenvalues of matrix $\mathbf{A}$, solve

$$\begin{vmatrix} \rho - \lambda & -\left(\frac{\rho+\delta}{\alpha} - \delta\right) \\ (\alpha-1)(\rho+\delta) & -\lambda \end{vmatrix} = 0$$

for λ and this gives

$$\lambda_1 = \frac{\rho - \sqrt{\rho^2 + 4(1-\alpha)(\rho+\delta)\left(\frac{\rho+\delta}{\alpha} - \delta\right)}}{2} < 0,$$
$$\lambda_2 = \frac{\rho + \sqrt{\rho^2 + 4(1-\alpha)(\rho+\delta)\left(\frac{\rho+\delta}{\alpha} - \delta\right)}}{2} > 0,$$

showing the existence of saddle path (λ_1 is the stable solution).

One can also arrive at the same conclusion in a simpler way. From

$$\mathbf{A} = \begin{bmatrix} \rho & -\left(\frac{\rho+\delta}{\alpha} - \delta\right) \\ (\alpha-1)(\rho+\delta) & 0 \end{bmatrix}$$

we know that

$$\lambda_1\lambda_2 = \det\mathbf{A} = \left(\frac{\rho+\delta}{\alpha} - \delta\right)(\alpha-1)(\rho+\delta) < 0,$$

(because $0 < \alpha < 1$, $\rho > 0$, $\delta > 0$) as well as

$$\lambda_1 + \lambda_2 = \text{tr}\mathbf{A} = \rho > 0.$$

Therefore

$$\Delta = \text{tr}^2\mathbf{A} - 4\det\mathbf{A} > 0$$

meaning that

- Matrix **A** has two different real eigenvalues ($\Delta > 0$);
- These two eigenvalues are different in signs ($\lambda_1\lambda_2 = \det\mathbf{A} < 0$);
- The system is unstable with a saddle path.

What's more, we can go one step further to calculate the *time paths*, i.e. the solutions for $\ln k(t)$ and $\ln c(t)$. From (2) we can find the solution for $\ln k(t)$

$$\ln k(t) = \ln k^* + c_1\alpha_1 \exp(\lambda_1 t) + c_2\beta_1 \exp(\lambda_2 t).$$

Given the fact that $\lambda_1 < 0$, $\lambda_2 > 0$ and $\lim_{t\to+\infty} \ln k(t) = \ln k^* < +\infty$, it must be that $c_2\beta_1 = 0$. And suppose that the system starts from the initial condition $\ln k(t = 0) = \ln k(0)$, then

$$\begin{aligned} \ln k(0) &= \ln k^* + c_1\alpha_1, \\ c_1\alpha_1 &= \ln k(0) - \ln k^*. \end{aligned}$$

Then the time path for $\ln k(t)$ is

$$\begin{aligned} \ln k(t) &= \ln k^* + [\ln k(0) - \ln k^*] \exp(\lambda_1 t), \\ \ln k(t) &= [1 - \exp(\lambda_1 t)] \ln k^* + \ln k(0) \exp(\lambda_1 t). \end{aligned}$$

And we do see that $\ln k(t)$ asymptotically converges to $\ln k^*$:

$$\lim_{t\to+\infty} \ln k(t) = \ln k^*.$$

Finding the time path for $\ln c(t)$ is left as an exercise for the readers.

3.3 ★ Systems of Higher Dimensions

Curious readers may wonder the same problem for the systems of higher dimensions, such as

$$\begin{cases} \dot{x}_1(t) = a_{11}x_1(t) + \ldots + a_{1n}x_n(t), \\ \dot{x}_2(t) = a_{21}x_1(t) + \ldots + a_{2n}x_n(t), \\ \ldots\ldots \\ \dot{x}_n(t) = a_{n1}x_1(t) + \ldots + a_{nn}x_n(t). \end{cases}$$

Write it in matrix form

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \cdots \\ \dot{x}_n(t) \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{1n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ \cdots \\ x_n(t) \end{bmatrix},$$

that is

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t).$$

Certainly there are similarities as the case for systems of lower dimensions. One can easily imagine that if all eigenvalues of $\mathbf{A}$ are positive, i.e. $\lambda_i \in \mathbb{R}_+, \forall i \in \{1, \ldots, n\}$, the system is unstable since for any point other than the steady state the forces from all directions push it away from the node. And by the similar argument, if all eigenvalues are positive, i.e. $\lambda_i < 0, \forall i \in \{1, \ldots, n\}$, the system is stable since for any point other than the steady state the forces from all directions push it towards the node.

The interesting question arises when m (with $1 \leq m < n$) eigenvalues are negative and $n - m$ are positive. The system is certainly not stable, however, one may ask whether there exists something like "saddle path"for two-dimension case following which the system is able to converge to the steady state. Theorem A.1 in the appendix gives the answer to this question. Basically such "saddle path"exists in the form of *m-dimensional manifold*, for example the saddle path in two-dimension case, a curve, is a one-dimensional manifold.

A more general conclusion is stated in Theorem A.2. Normally numerical methods with the help of computers are the only choice when people analyze high dimensional systems. Interested readers may refer to Zeidler *et al.* (2003).

4 Readings

Barro and Sala-í-Martin (2004), Appendix of Mathematical Methods A.1.

5 Bibliographic Notes

Example in Section 2 is a simplified exposure of Romer (2006) Chapter 2A, and the example in Section 3.2.2 is similar to that in Barro and Sala-í-Martin (2004), Chapter 2 Appendix.

Methods of analyzing systems of differential equations can be found in any one of the mathematical books for economists listed in the first chapter of our class notes, and Barro and Sala-í-Martin (2004), Appendix of Mathematical Methods A.1 presents a concise summary for your quick references. For one who wants more details, Blanchard, Devaney and Hall (1998) is recommended for its completeness and accessibility.

6 Exercises

6.1 Graphical Analysis: Phase Diagram

a)$^{\mathbf{A}}$ Find the solution of the following initial value problem. Analyze the dynamics in the phase space, and describe the behavior of the solution as $t \to +\infty$.

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} = \begin{bmatrix} -2 & 1 \\ -5 & 4 \end{bmatrix} \mathbf{x} \text{ with } \mathbf{x}(0) = \begin{bmatrix} 1 \\ 3 \end{bmatrix}.$$

b)$^{\mathbf{B}}$ Consider each of the following systems of ordinary differential equations. Analyze the dynamics in the phase space, and describe the stability properties of the steady-state(s).

1. $$\begin{cases} \dot{x}_1 = \frac{1}{2}x_1^{\frac{1}{2}} - \frac{1}{2}x_1 - x_2, \\ \dot{x}_2 = x_1^{-\frac{1}{2}} - 1. \end{cases}$$

2. $$\begin{cases} \dot{x}_1 = \frac{1}{2}x_1^{\frac{1}{2}} - \frac{1}{2}x_1 - x_2, \\ \dot{x}_2 = x_2 - x_1 \end{cases} \quad \text{with } x_1 \neq 0.$$

6.2 Analytical Analysis$^{\mathbf{B}}$

Redo Problem 6.1 **b)** with analytical approach.

References

Barro, R. J. and X. Sala-í-Martin (2004): *Economic Growth (2nd Ed.)*. Cambridge: MIT Press.

Blanchard, P., Devaney, R. L. and G. R. Hall (1998): *Differential Equations*. California: Pacific Grove.

Romer, D. (2006): *Advanced Macroeconomics (3rd Ed.)*. Boston: McGraw-Hill Irwin.

Zeidler, E. , Grosche, G. and I. N. Bronstein (Hrsg.) (2003): *Teubner – Taschenbuch der Mathematik 2 (8. Aufl.)* . Stuttgart: Teubner Verlag.

Appendix

A Useful Results of Mathematics

A.1 Definitions of Stability

Definition *(Lyapnov Stability)* For a system of ordinary differential equations

$$\dot{\mathbf{x}}(t) = \mathbf{F}(\mathbf{x}(t))$$

a state $\mathbf{x}^*$ is *Lyapunov stable* if every neighbourhood $B_{\mathbf{X}^*}$ of $\mathbf{x}^*$ contains a neighbourhood $B_0 \subseteq B_{\mathbf{X}^*}$ of $\mathbf{x}^*$ such that given the initial condition $\mathbf{x}(0)$ the solution to the system lies in $B_{\mathbf{X}^*}$, $\forall \mathbf{x}(0) \in B_0$.

Simply speaking, Lyapnov stability implies that given a small perturbation on the state $\mathbf{x}^*$ the system doesn't move *further away* from $\mathbf{x}^*$ with the time going on.

Definition *(Asymptotic Stability)* For a system of ordinary differential equations

$$\dot{\mathbf{x}}(t) = \mathbf{F}(\mathbf{x}(t))$$

a state $\mathbf{x}^*$ is *asymptotically stable* if it is Lyapunov stable and $\exists B^*_{\mathbf{X}^*}$ such that $\forall \mathbf{x}(0) \in B^*_{\mathbf{X}^*}$

$$\lim_{t\to+\infty} \mathbf{x}(t)|_{\mathbf{X}(t=0)=\mathbf{X}(0)} = \mathbf{x}^*.$$

Simply speaking, asymptotic stability implies that given a small perturbation on the state $\mathbf{x}^*$ a Lyapnov stable system *move back towards* $\mathbf{x}^*$ with the time going on.

A.2 General Theorems for the Path of Convergence

Theorem A.1 (Path of Convergence, Linear Differential Equation System) *Consider the following linear differential equations system*

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{b}$$

with initial value $\boldsymbol{x}(0)$*, where* $\boldsymbol{x}(t) \in \mathbb{R}^n$ *for all* t *and* $\boldsymbol{A}$ *is an* $n \times n$ *matrix. Suppose that* $\boldsymbol{x}^*$ *is the steady state of the system, i.e.* $\boldsymbol{A}\boldsymbol{x}^* + \boldsymbol{b} = \boldsymbol{0}$*. Suppose that* $m \leq n$ *of the eigenvalues of* $\boldsymbol{A}$ *have negative real parts. Then there exists an m-dimensional manifold* $\mathcal{M}$ *of* $\mathbb{R}^n$ *such that starting from any* $\boldsymbol{x}(0) \in \mathcal{M}$*, the differential equation has a unique solution with* $\boldsymbol{x}(t) \to \boldsymbol{x}^*$*.*

Theorem A.2 (Path of Convergence, General Differential Equation System) *Consider the following general autonomous differential equations system*

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{F}(\boldsymbol{x}(t))$$

in which $\boldsymbol{F} : \mathbb{R}^n \to \mathbb{R}^n$ *and suppose that* $\boldsymbol{F}$ *is continuously differentiable, with initial value* $\boldsymbol{x}(0)$*. Suppose that* $\boldsymbol{x}^*$ *is the steady state of the system, i.e.* $\boldsymbol{F}(\boldsymbol{x}(t)) = \boldsymbol{0}$*. Define*

$$\boldsymbol{A} = \nabla \boldsymbol{F}(\boldsymbol{x}^*)$$

and suppose that $m \leq n$ *of the eigenvalues of* $\boldsymbol{A}$ *have negative real parts and the rest have positive real parts. Then there exists an open neighborhood of* $\boldsymbol{x}^*$*,* $\boldsymbol{B}_{\boldsymbol{x}^*} \in \mathbb{R}^n$ *and an m-dimensional manifold* $\mathcal{M} \in \boldsymbol{B}_{\boldsymbol{x}^*}$ *such that starting from any* $\boldsymbol{x}(0) \in \mathcal{M}$*, the differential equation has a unique solution with* $\boldsymbol{x}(t) \to \boldsymbol{x}^*$*.*