1

Think as a Macroeconomist: Micro Behaviour and Macro Modelling

Nietzsche ist tot.

—Gott

My nearly two decades as a researcher in a central bank taught me two lessons. First, formal analysis, both theoretical and applied, can and does make an immense contribution to improving the conduct of policy. Second, fundamental advances over the last 20 years — say, in basic econometric techniques, in modelling dynamic systems in which expectations are important, and in data sources — provide an opportunity for historically rapid advance in our understanding of macroeconomic policymaking over the coming years.

—Jon Faust

1 Introduction

This is a warming-up session for our first lecture of *Macroeconomics (Research)*, helping the readers get used to the basic modelling ideas in macro.

Modern macroeconomics is strongly featured by its sound foundation of the micro behaviours. Its methodology is based on the dynamic general equilibrium models, with the potential to accommodate stochastic environments. Therefore the central issue in this chapter is to explain how to model the micro behaviours of the economic agents, the concepts of which are already familar for the readers, in the macro (dynamic) set-ups. In the end we apply these settings in a simple partial equilibrium model, and attempt to obtain some flavour of dynamic equilibrium modellings.

2 Economic Agents and Decision Making Problems

Mostly concerned agents in macroeconomic models are households and firms. The former offer labor to earn wage income, and make decisions in consumption and accumulation of wealth. The latter produce consumption goods by employing labor and investing in capital stock. Therefore to study the behaviour of agents in the economy provides important insights in understanding macroeconomic phenomenen.

2.1 Households

2.1.1 Preferences

As you have already learned in intermediate microeconomic theory, the rational preference of an economic agent can be captured by a utility function

$$u(\mathbf{x}) : \mathbb{R}^n \to \mathbb{R}$$

in which $\mathbf{x} = (x_1, \ldots, x_n)$ is the bundle containing n goods that gives utility to the agent. For simplicity we assume that the utility function is well-behaved with the following properties (which are actually more than sufficient for the problems in macroeconomics)

$$\frac{\partial u}{\partial x_i} > 0, \frac{\partial^2 u}{\partial x_i^2} < 0, \forall i \in \{1, \ldots, n\},$$

i.e. u is strictly concave in all x_i's.

Now in a dynamic context we have to study the agent's utility function over a period of time. The simplest way to do that is the approach that you learned in intermediate macroeconomic theory. Suppose the time is devided into $n + 1$ *discrete* periods, and in each interim period $t \in \{0, 1, \ldots, n\}$ the agent gains her utility from the bundle $\mathbf{x}_t$ in this period. Then the utility function over the entire time span can be expressed as a summation

$$
\begin{aligned}
U &= \sum_{t=0}^{n} \beta^t u(\mathbf{x}_t) \\
&= \sum_{t=0}^{n} \left(\frac{1}{1+\rho}\right)^t u(\mathbf{x}_t)
\end{aligned}
$$

in which β or $\frac{1}{1+\rho}$ act as *discount factors*. This approach implies that the utility function is additive with

$$
\frac{\partial^2 U}{\partial x_{it} \partial x_{it'}} = 0, \forall t \neq t'.
$$

Although such discrete-time approach seems straight-forward for understanding, in practice the other treatment, continuous-time approach, sometimes provides better computational tractability. Now if we split T into more periods, i.e. we increase n towards infinity, then each period is so small that the time becomes *continuous*. Therefore when n is infinite the summation above becomes an integration

$$
U = \int_0^T e^{-\rho t} u(\mathbf{x}(t)) dt
$$

giving a utility function in continuous time. Since readers are less likely to have sufficient experience with the continuous-time models, the following sections are mostly written in a continuous-time manner. The results from the discrete-time approach are summerized in the end of this chapter, and the developments of these results are left as your exercises.

2.1.2 Intertemporal Resource Constraints

Households are subject to the resource constraints, meaning that they cannot always consume as much as they want. Suppose that one household is established when $t = 0$ with population $L(0)$ and initial assets $A(0)$ [1] . The population grows at a rate of n, and at time $T >> 0$ the household is dissolved. Nobody dies for $t \in [0, T]$.

[1] Conventionally people use capital letter for aggregate value, and small letter for per capita value. What's more, people use small letter with hat, e.g. $\hat{k}$, to denote the value for per effective labor. We will see this in Section 3.

There are following resources the household is able to manage during its life span:

- Everybody begins to work immediately after birth and never stops working before T, receiving a wage $w(t)$ for each moment $t \in (0, T)$;
- The household has a flow of consumption $C(t)$ for $t \in (0, T)$;
- The household rents its assets in an exogenous assets market, and at each moment $t \in (0, T)$ it gets a return $r(t)A(t)$ given the market rate $r(t)$ and the level of the assets holding $A(t)$ at this moment [2] ;
- And one has to keep in mind that since the population of the household grows at a rate n, per capita assets are actually shrinking at the same rate because the household's assets have to be equally distributed among all its members. We will see this effect later.

One can think about the household's resource constraint in two ways:

- From its life-time point of view what it consumes in its life cannot exceed what it earns. Therefore we can express its *life-time budget constraint* as

$$\int_0^T \exp\left(-\int_0^t r(s)ds\right) C(t)dt \leq A(0) + \int_0^T \exp\left(-\int_0^t r(s)ds\right) w(t)L(t)dt. \tag{1}$$

  Note that the inequality above is expressed as present value at $t = 0$;
- From an instantaneous point of view, at any moment $t \in (0, T)$ what it earns less what it loses becomes the increment in its assets holding. Therefore we can express its *flow budget constraint* (also called *law of motion*) as

$$\dot{A}(t) \leq w(t)L(t) + r(t)A(t) - C(t) \tag{2}$$

  with the boundary conditions

$$A(t = 0) = A(0),$$
$$\exp\left(-\int_0^t r(s)ds\right) A(t = T) \geq 0.$$

The latter says that the household is not allowed to end up with strictly positive debt.

It's fairly trivial to write down such budget constraints in the household level. However, to stand in line with our representative agent argument we have to adapt these constraints into the individual level, i.e. in per capita terms, and this is less trivial to see. Now from (1) using the fact that the population grows exponentially, $L(t) = L(0)e^{nt}$, we can express the household's life-time budget constraint in per capita terms

[2] This assumption already integrated the possibility that the household may borrow via debt contract – debts can simply be treated as negative assets with the same market rate $r(t)$.

$$
\begin{aligned}
\int_0^T \exp\left(-\int_0^t r(s)ds\right) c(t)L(0)e^{nt}dt &\le a(0)L(0) + \int_0^T \exp\left(-\int_0^t r(s)ds\right) w(t)L(0)e^{nt}dt, \\
\int_0^T \exp\left(-\int_0^t [r(s)-n]ds\right) c(t)dt &\le a(0) + \int_0^T \exp\left(-\int_0^t [r(s)-n]ds\right) w(t)dt.
\end{aligned}
$$

From (2) we can also express the household's flow budget constraint in per capita terms. Note that using log-linearization

$$
\begin{aligned}
&A(t) = a(t)L(t), \\
&\frac{\dot{A}(t)}{A(t)} = \frac{\dot{a}(t)}{a(t)} + \frac{\dot{L}(t)}{L(t)}, \\
&\dot{a}(t) = \frac{\dot{A}(t)}{L(t)} - na(t).
\end{aligned}
$$

Insert the last equation into (2) and get

$$
\begin{aligned}
&\dot{A}(t) \le w(t)L(t) + r(t)A(t) - C(t), \\
&\frac{\dot{A}(t)}{L(t)} \le w(t) + r(t)\frac{A(t)}{L(t)} - \frac{C(t)}{L(t)}, \\
&\dot{a}(t) \le w(t) + r(t)a(t) - na(t) - c(t).
\end{aligned}
$$

with the boundary conditions

$$
\begin{aligned}
a(t=0) &= a(0), \\
\exp\left(-\int_0^t [r(s)-n]ds\right) a(t=T) &\ge 0.
\end{aligned}
$$

For the budget constraints in either household level or individual level, the two types of expressions, i.e. life-time constraints and flow constraint, seem to be quite different from each other in forms, although the reasoning behind both sounds almost equally undoubtable. But do they really contain the same information, or, are they really interchangeable when they appear in optimization problems as budget constraints? Well, if $T < +\infty$, i.e. if the end point T is finite time, these two expressions are equivalent. But people do have to worry when T goes to infinity (in this case there is no end-point boundary condition). We will see the deep reason behind this issue in the later lectures, and readers can already find the lengthy mathematical argument in one example of Appendix A.2.4.

Writing down the right intertemporal resource constraints is very important in solving dynamic problems. We will see many similar constraints later concerning different kinds of agents (consumers, firms, governments. . .) in different settings (discrete or continuous time, models with money, debt, bonds, international trade, etc.), and one really has to look carefully into the details of the timing structures in order to make everything correct. In addition one has to be careful when adapting the aggregate resource constraints into the individual ones (per capita forms).

2.2 Firms

Now let's have a look of the firms' problem. Shortly speaking a firm in an economy arranges its production with a certain technology in order to maximize its profit. The following sections discuss how to model these feasibilities and motivations.

2.2.1 Technology

The firms adopt the technology described by the *neoclassical production function* F with capital K[3] and labor L as input, i.e. output $Y = F(K, L) : \mathbb{R}^2 \to \mathbb{R}$. The production function is neoclassical in the sense that it fulfills the following (relatively mild) assumptions:

(1) **Constant return to scale (CRS)** If we replicate a factory by doubling the capital and labor input, the output is also doubled:

$$F(\lambda K, \lambda L) = \lambda F(K, L), \forall \lambda \in \mathbb{R}_{++}.$$

Therefore $F(K, L)$ is homogenous of degree one in K and L.

(2) **Diminishing marginal return** $\forall K, L \in \mathbb{R}_{++}$ $F(K, L)$ exhibits

$$\frac{\partial F}{\partial K} > 0, \frac{\partial^2 F}{\partial K^2} < 0,$$
$$\frac{\partial F}{\partial L} > 0, \frac{\partial^2 F}{\partial L^2} < 0.$$

(3) **Inada conditions** We'll see the importance of the conditions later.

$$\lim_{K\to 0} \frac{\partial F}{\partial K} = \lim_{L\to 0} \frac{\partial F}{\partial L} = +\infty,$$
$$\lim_{K\to +\infty} \frac{\partial F}{\partial K} = \lim_{L\to +\infty} \frac{\partial F}{\partial L} = 0.$$

These three assumptions directly lead to the following property of neoclassical production functions:

[3] Please keep in mind that K and L are functions of time t, i.e. $K(t)$ and $L(t)$. We drop t where it doesn't lead to confusions.

Proposition 2.1 ***(Essentiality)*** *Each input is essential in the sense that*

$$F(0, L) = F(K, 0) = 0.$$

Proof As a first step, it's easy to see that

$$\lim_{K\to+\infty} \frac{F}{K} = \lim_{K\to+\infty} \frac{\frac{\partial F}{\partial K}}{1} = 0$$

by (1) L'Hôpital rule (Why is L'Hôpital rule plausible here, i.e. why $\lim_{K\to+\infty} F(K, L) = +\infty$?) and (2) Inada condition. Then

$$\lim_{K\to+\infty} \frac{F}{K} = \lim_{K\to+\infty} F\left(1, \frac{L}{K}\right) = F(1, 0) = 0$$

by constant return to scale. And

$$F(K, 0) = KF(1, 0) = 0$$

by applying again the assumption of constant return to scale. Similar argument holds for proving $F(0, L) = 0$. □

2.2.2 Profit Maximization

As we learned in intermediate micro the firms maximize their profit by

$$\max_{K,L} \Pi = F(K, L) - rK - \delta K - wL$$

in which r is the real interest rate, δ is the depreciation rate of the capital, and w is the wage rate for employees. Therefore, $R = r + \delta$ defines the rental rate for the firms to get the capital. Suppose that firms obtain capital and labor from corresponding competitive markets in which r and w are determined as equilibrium prices.

The first order conditions require that [4]

[4] For a multi-variate function $F(x_1, \ldots, x_n) : \mathbb{R}^n \to \mathbb{R}$, as a convention in economics, people often denote its partial derivative with respect to the k-th ($1 \le k \le n$) variable by $F_{x_k} := \frac{\partial F(x_1,\ldots,x_n)}{\partial x_k}$, or $F_k := \frac{\partial F(x_1,\ldots,x_n)}{\partial x_k}$. For example, people write $\frac{\partial F}{\partial K}$ as F_K or F_1 (and $\frac{\partial^2 F}{\partial K \partial L}$ as F_{KL} or F_{12}). In these class notes the author prefers to write all partial derivatives in explicit forms. However, readers are asked to get used to this convention while reading the other literature.

$$\frac{\partial \Pi}{\partial K} = \frac{\partial F}{\partial K} - r - \delta = 0,$$
$$\frac{\partial \Pi}{\partial L} = \frac{\partial F}{\partial L} - w = 0$$

meaning that the firms set the input level of capital exactly at which the marginal product of capital is equal to the marginal cost, and the input level of labor is exactly set at the point where the marginal product of labor is equal to the marginal cost.

Again as we did in the section for households the problems concerning firms can also be expressed in per capita terms. For example per capita capital intensity $k = \frac{K}{L}$, as well as per capita output

$$y = \frac{Y}{L} = \frac{F(K, L)}{L} = F\left(\frac{K}{L}, 1\right) = F(k, 1)$$

from the fact that $F(K, L)$ is homogenous of degree 1. And $F(k, 1)$ simply means the output generated by per capita capital input and per capita labor input, which is defined as the production function in per capita form

$$y = f(k) = F(k, 1).$$

As an exercise readers can verify that $f(k)$ has the neoclassical properties such that

- **Constant return to scale (CRS)**

  $$f(\lambda k) = \lambda f(k), \forall \lambda \in \mathbb{R}_{++}.$$

  Therefore $f(k)$ is homogenous of degree one in k.
- **Diminishing marginal return** $\forall k \in \mathbb{R}_{++}$, $f(k)$ exhibits

  $$f'(k) > 0, f''(k) < 0.$$

- **Inada conditions** If $F(K, L)$ fulfills Inada condition, then $f(k)$ fulfills as well.

  $$\lim_{k \to 0} f'(k) = +\infty,$$
  $$\lim_{k \to +\infty} f'(k) = 0.$$

Rewrite the first order conditions in terms of per capita variables

$$\begin{aligned} r &= \frac{\partial F}{\partial K} - \delta \\ &= \frac{\partial F}{L \partial k} - \delta \\ &= f'(k) - \delta, \end{aligned}$$

and by the Euler's formula

$$F(K,L) = \frac{\partial F}{\partial K}K + \frac{\partial F}{\partial L}L,$$
$$f(k) = kf'(k) - w,$$

one can see that

$$w = f(k) - kf'(k).$$

2.3 Technological Progress

Technological progress is a huge dimension in considering sustainable economic growth. The simplest way (as we do for most of the time in this course) to integrate technological progress into a macro model is to modify the production function – basically one can easily imagine the following three possible treatments:

(1) We assume that technological progress helps to magnify the output in the form of *Hicks-neutral* production function

$$Y(t) = A(t)F\left(K(t), L(t)\right)$$

in which $A(t) \in \mathbb{R}_{++}, \forall t$ is the function capturing technological progress;

(2) We assume that technological progress helps to save the input of capital in the form of *Solow-neutral* (or *capital-augmenting*) production function

$$Y(t) = F\left(A(t)K(t), L(t)\right);$$

(3) We assume that technological progress helps to save the input of labor in the form of *Harrod-neutral* (or *labor-augmenting*) production function

$$Y(t) = F\left(K(t), A(t)L(t)\right).$$

However people mostly use the last form (labor-augmenting production function) in practices. The deep reason is that for most of the settings this is the only form that ensures the existence of a steady state in dynamic analysis. See Uzawa (1961) and Schlicht (2006).

3 A Simple Dynamic Partial Equilibrium Model (Solow-Swan Model)

Solow (1957) and Swan (1956) consider an economy with exogenous saving rate $s \in [0, 1]$, and all the other parameters are endogenously determined – that's why it is a partial equilibrium model. Suppose that at each moment t, $Y(t) = F\left(K(t), A(t)L(t)\right)$ output is produced via a neoclassical, labor augmenting production function. A share $1 - s$ of $Y(t)$ is consumed as $C(t) = (1 - s)Y(t)$, and s of $Y(t)$ is saved as investment. Moreover,

- The depreciation rate for capital stock $K(t)$ is δ;
- Technological progress index $A(t)$ grows at a rate g;
- Population grows at a rate n.

In the end what is left in this economy becomes the change in capital stock:

$$\begin{aligned}\dot{K}(t) &= I(t) - \delta K(t),\\ &= F\left(K(t), A(t)L(t)\right) - C(t) - \delta K(t),\\ &= sF\left(K(t), A(t)L(t)\right) - \delta K(t).\end{aligned}$$

From $\hat{k}(t) = \frac{K(t)}{A(t)L(t)}$ by log-linearization

$$\begin{aligned}\frac{\dot{\hat{k}}(t)}{\hat{k}(t)} &= \frac{\dot{K}(t)}{K(t)} - \frac{\dot{A}(t)}{A(t)} - \frac{\dot{L}(t)}{L(t)},\\ \dot{\hat{k}}(t) &= \frac{\dot{K}(t)}{K(t)}\frac{K(t)}{A(t)L(t)} - (n+g)\hat{k}(t)\\ &= \frac{\dot{K}(t)}{A(t)L(t)} - (n+g)\hat{k}(t),\end{aligned}$$

then insert the expression for $\dot{K}(t)$

$$\begin{aligned}\dot{\hat{k}}(t) &= \frac{sF\left(K(t), A(t)L(t)\right) - \delta K(t)}{A(t)L(t)} - (n+g)\hat{k}(t),\\ &= sf\left(\hat{k}(t)\right) - (\delta + n + g)\hat{k}(t).\end{aligned} \tag{3}$$

Next, as Romer (2006), one can analyze the economic dynamics in a graphical approach, as Figure 1 shows, with the steady state value $\hat{k}^*$ as the economy's long-run equilibrium. However, merely with function $sf\left(\hat{k}(t)\right)$ being concave, there may be cases other than Figure 1. For example, in Figure 2 (a) the curve $sf\left(\hat{k}(t)\right)$ lies below $(\delta + n + g)\,\hat{k}(t)$, and in Figure 2 (b) the curve $sf\left(\hat{k}(t)\right)$ converges to a line paralell to $(\delta + n + g)\,\hat{k}(t)$ — in these two cases, there exists no steady state $\hat{k}^* > 0$.

The dynamic system is made deterministic by adding Inada conditions, which ensures a unique steady state $\hat{k}^* > 0$. To show this, rewrite equation (3) as

$$\frac{\dot{\hat{k}}(t)}{\hat{k}(t)} = \frac{sf\left(\hat{k}(t)\right)}{\hat{k}(t)} - (\delta + n + g).$$

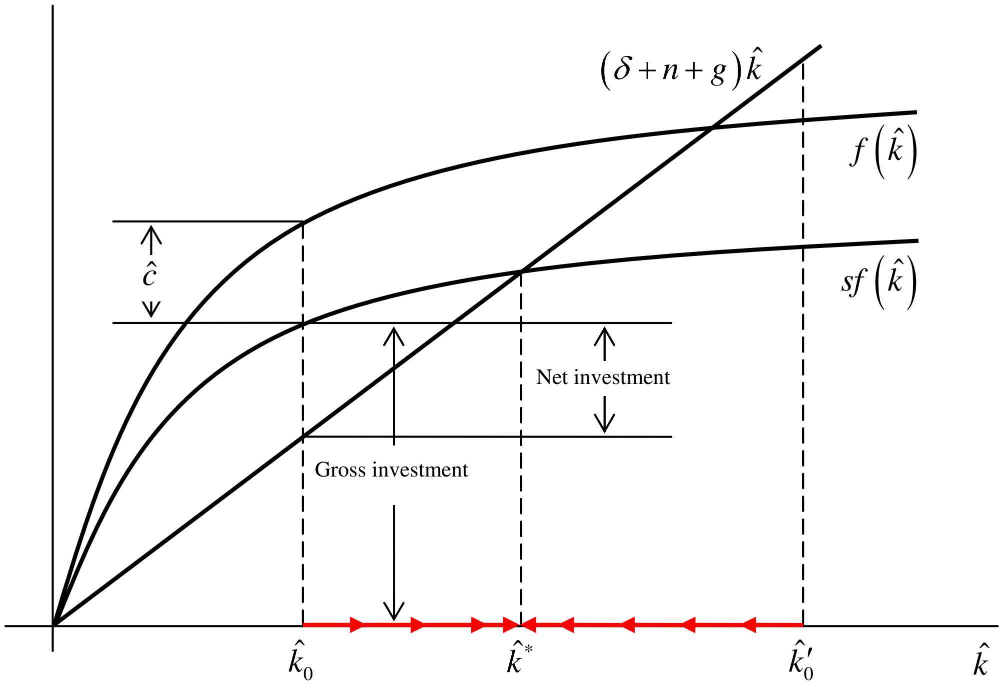

Fig. 1. The Dynamics of k

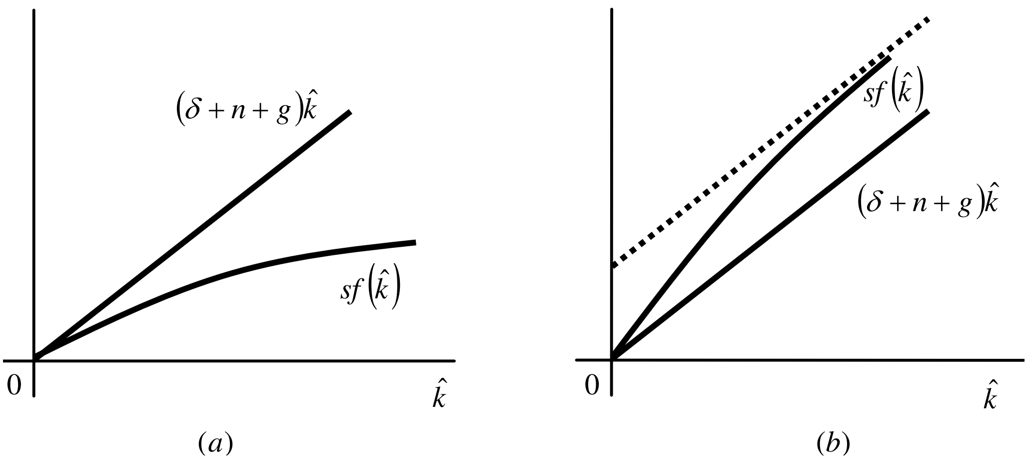

Fig. 2. Other Cases of Convergence

Then we claim that

(1) Function $\frac{sf(\hat{k}(t))}{\hat{k}(t)}$ is monotonically decreasing with $\hat{k}(t)$ (Why?);

(2) By Inada conditions, $\lim_{\hat{k}(t)\to 0} \frac{sf(\hat{k}(t))}{\hat{k}(t)} = +\infty$ and $\lim_{\hat{k}(t)\to +\infty} \frac{sf(\hat{k}(t))}{\hat{k}(t)} = 0$ (Why?);

(3) By claims 1 and 2, there exists a unique $\hat{k}^* > 0$ such that $\frac{\dot{\hat{k}}(t)}{\hat{k}(t)} = 0$ as Figure 3 shows (Why?).

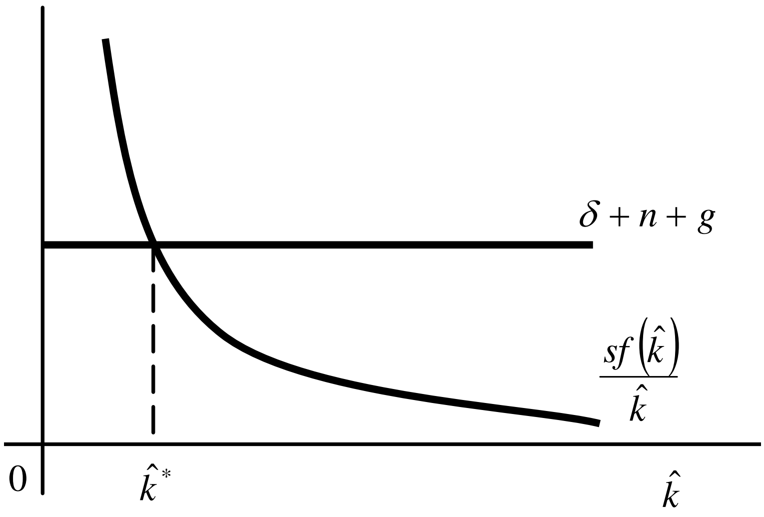

Fig. 3. Convergence under Inada Condition

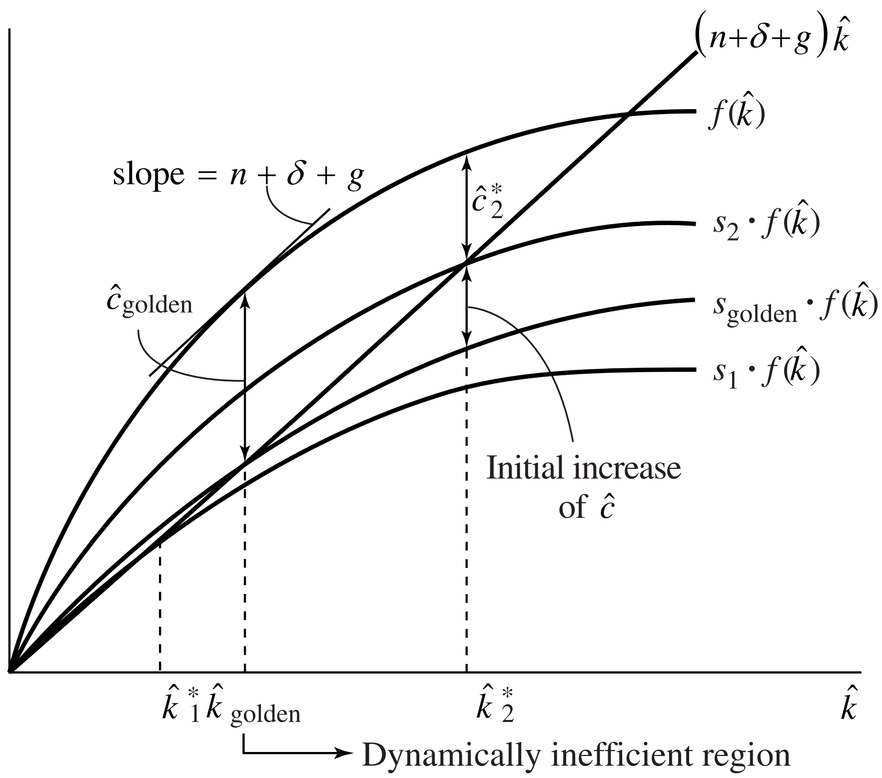

Fig. 4. The Golden Rule (*Stolen from Barro and Sala-í-Martin (2004), p. 36*)

Further Issues (To be discussed in the class):

- Dynamics and path of convergence (Figure 1)

- Golden rule and dynamic (in-)efficiency (FIGURE 4)
- Stability
- Speed of convergence
- Comparative statics
- Continuous- versus Discrete-time approach

4 Readings

Romer (2006) Chapter 1, *or* Barro and Sala-í-Martin (2004) Chapter 1.

5 Bibliographic Notes

Most of the material presented in this lecture can be found in the introductory chapters of every textbook on advanced macroeconomics or economic growth, e.g. Acemoglu (2009), Barro and Sala-í-Martin (2004), Romer (2006), Solow (2000), just name a few. Further discussions on microeconomic theory can be found in the classics like Mas-Colell *et al.* (1995), Varian (1992), or the soon-to-be classic Rubinstein (2006).

Prerequiste of analytical techniques in this course is first-year undergraduate mathematics, roughly equivalent to Strang (1991) which includes essentials of calculus, basic knowledge of linear algebra as well as probability and statistics (as Hogg *et al.* (2004)). In the appendix of each chapter readers find fundamental mathematical facts which one may need to read through the text.

There are many handbooks from which reader may get some quick references. Economists may prefer the special design of Sydsæter, Strøm and Berck (2005). For one who wants to go into technical details, there is a pretty wide collection of excellent textbooks for economic analysis, such as (at an increasing level of difficulty and analytical rigor) Hoy *et al.* (2001), Sydsæter, and Hammond (2005), Chiang *et al.* (2005), Simon *et al.* (1994), de la Fuente (2000) and Sydsæte, Hammond, Seierstad and Strøm (2005). Above all, Kolmogorov and Fomin (1970), Rudin (1976) provide readers relatively easier entrance to advanced mathematical analysis.

6 Exercises

6.1 Solow-Swan Model

As in the standard Solow-Swan model, assume that both labor and capital are paid their marginal products and the production follows labor-augmenting neoclassical production function. Let w denote $\frac{\partial F(K,AL)}{\partial L}$ and r denote $\frac{\partial F(K,AL)}{\partial K} - \delta$.

a)$^{\mathbf{A}}$ [5] Show that the marginal product of labor is $w = A[f(k) - kf'(k)]$.

b)$^{\mathbf{A}}$ Show that if both capital and labor are paid their marginal products, constant return to scale imply that the total amount paid to the factors of production equals total net output, i.e. $wL + rK = F(K, AL) - \delta K$.

c)$^{\mathbf{B}}$ What are the growth rates of r and w on a balanced growth path? Show that this model exhibits the properties as *Kaldor facts*, such that r is roughly constant over time, as are the shares of output going to capital and to labor.

d)$^{\mathbf{B}}$ Suppose that the economy starts with a level of k less than k^*. As the time going on, is w growing at a rate greater than, less than, or equal to its growth rate on the balanced growth path? What about r?

6.2 Solow-Swan Model with Human Capital

(Hall & Jones, 1999) Suppose, similar as standard Solow-Swan model, in an economy with constant population growth rate $\frac{\dot{L}}{L} = n$ as well as constant exogenous technological progress rate $\frac{\dot{A}}{A} = g$, the output Y is produced with physical capital K and human capital H, according to a Cobb-Douglas technology

$$Y = K^{\alpha}(AH)^{1-\alpha}, 0 < \alpha < 1.$$

Human capital is accumulated by workers (raw labor, L) by investing into education or training. Suppose that individuals spend a constant fraction of their time, u, learning and that skills are accumulated according to the following expression

$$H = e^{\psi u}L,$$

where ψ measures the percentage increase in H following a small increase in u, i.e. $\frac{d\ln H}{du} = \psi$. (Note that if $u = 0$, $H = L$ and all labour is unskilled). The rest of the economy is as in the standard Solow-Swan model. Physical capital obeys the law of motion

[5] The level of difficulty. A is the lowest.

$$\dot{K} = sY - \delta K.$$

a)$^{\mathbf{A}}$ Express the production function in terms of output per effective worker.

b)$^{\mathbf{A}}$ What is the fundamental equation of motion for this economy? Illustrate the dynamic behaviour of the system using a diagram.

c)$^{\mathbf{A}}$ Find the steady state values of output per effective worker and output per capita.

d)$^{\mathbf{B}}$ Suppose a country decides to increase the fraction of time devoted to education, u. What will happen to output per capita in steady state, $\frac{\partial y^*}{\partial u}$?

e)$^{\mathbf{A}}$ Show the transition to the new steady state in the phase diagram.

6.3 Solow-Swan Model with Minimum Wage

Consider a Solow-Swan economy with firms adopting Cobb-Douglas technology

$$F(K(t), L(t)) = K(t)^{\alpha}(A(t)L(t))^{1-\alpha}$$

in which the index of technological progress $A(t)$ grows at a rate g, starting from $A(0)$. Suppose that the capital depreciates at a rate δ and the labor force grows at a rate n. The saving rate is constant s for all the time.

a)$^{\mathbf{A}}$ Write down the capital flow in terms of per capita variables.

b)$^{\mathbf{B}}$ The economy starts with per effective labor capital $k(0)$. Calculate $k(t)$ and show that

$$\lim_{t \to +\infty} k(t) = k^*$$

in which k^* is the steady state level of capital stock per effective labor.

c)$^{\mathbf{B}}$ What fraction of growth in $\frac{Y}{L}$ does the growth accounting framework attribute to growth in $\frac{K}{L}$? What fraction to technological progress?

d)$^{\mathbf{A}}$ How can you reconcile this finding with the fact that the Solow model implies that the growth rate of $\frac{Y}{L}$ on the balanced growth path is solely determined by the exogenous rate of technological progress?

e)$^{\mathbf{B}}$ If we increase the saving rate s by ten percent, how will the steady state output per effective labor, $f(k^*)$, change?

f)$^{\mathbf{C}}$ Show that under constant saving rate s the steady-state *per capita* real wage and consumption grows at rate g. Now suppose that an economy is already in the steady state in $t = T$.

The Labor Party proposes the introduction of Minimum Wage Act concerning a per capita wage increase at T, $\underline{w} > w_T$, and from T onwards it grows exponentially at rate g (w_T is the steady-state value of wage rate at T). Characterize the evolution of employment, capital, and output for all $t > T$ under the following two different proposals:

(1) The Act is effective forever;
(2) The Act is effective till $t = T' > T$. And then the minimum wage is adjusted to a new growth rate, i.e. the new minimum wage is defined as $\underline{w'}(t)$ which grows from $\underline{w}_{T'}$ at a rate $0 < g' < g$ ($\underline{w}_{T'}$ is the previous period stipulated minimum wage level at T'). Show that the minimum wage growing at rate g' initially slows down the rise in unemployment and later on leads to increasing levels of employment until full employment is reestablished. Argue that at this date the minimum wage ceases to be binding and that the actual wage per effective labor as well as the capital stock per effective labor is lower than their initial laissez-faire level.

6.4 Solow-Swan Model with Endogenous Labor Force Participation

Consider the standard Solow-Swan model in which the population L_t grows at a constant exponential rate, i.e. $\frac{\dot{L}_t}{L_t} = n_L$. Abstract from technological progress and assume that the labor force participation rate is a function of the real wage rate w_t, according to

$$p(w_t) = \frac{N_t}{L_t},$$

where N_t is employment. Assume that the production function is Cobb-Douglas,

$$Y_t = K_t^{\alpha} N_t^{1-\alpha}, 0 < \alpha < 1.$$

a)$^{\mathbf{C}}$ Develope the fundamental differential equation for the per capita capital stock $k_t = \frac{K_t}{L_t}$ and show tha it depends on the elasticity of the participation rate with respect to the wage η_{pw} and on the elasticity of wages with respect to per capita capital η_{wk}.

b)$^{\mathbf{A}}$ What are the likely signs of η_{pw} and η_{wk}? Explain intuitively.

c)$^{\mathbf{B}}$ Explain both formally and intuitively what the effect of an endogenous participation rate is on the adjustment speed of the economy.

6.5 Solow-Swan Model with Endogenous Heterogeneity in Technology Diffusion

Instead of Solow-Swan model in an individual economy, suppose that the world economy consists of J countries, indexed $j = 1, ..., J$, each with access to a neoclassical aggregate production function for producing a unique final good,

$$Y_j(t) = F\left(K_j(t), A_j(t)L_j(t)\right),$$

where $Y_j(t)$ is the output of this unique final good in country j at time t, and $K_j(t)$ and $L_j(t)$ are the capital stock and labor supply. Finally, $A_j(t)$ is the technology of this economy, which is both country-specific and time-varying. To ease our discussions in the following, define per capita income as well as the effective capital-labor ratio in country j at time t as

$$y_j(t) = \frac{Y_j(t)}{L_j(t)},$$
$$k_j(t) = \frac{K_j(t)}{A_j(t)L_j(t)}.$$

Suppose that time is continuous, that there is population growth at the constant rate $n_j \geq 0$ in country j, and that there is an exogenous saving rate equal to $s_j \in (0, 1)$ in country j and a depreciation rate of $\delta \geq 0$ for capital. Define the growth rate of technology of country j at time t as

$$g_j(t) = \frac{\dot{A}_j(t)}{A_j(t)},$$

and the initial conditions are $k_j(0) > 0$ and $A_j(0) > 0$ for each $j = 1, ..., J$.

a)$^{\mathbf{A}}$ Derive the law of motion of $k_j(t)$ for each country.

Now let us assume that the worlds *technology frontier*, denoted by $A(t) = \max\{A_1(t), ..., A_J(t)\}$, grows exogenously at the constant rate

$$g(t) = \frac{\dot{A}(t)}{A(t)} > 0$$

with an initial condition $A(0) > 0$. Moreover, each countrys technology progresses as a result of absorbing the worlds technological knowledge. In particular, let us posit the following law of motion for each countrys technology:

$$\dot{A}_j(t) = \sigma_j\left(A(t) - A_j(t)\right) + \lambda_j A_j(t)$$

where $\sigma_j \in (0, +\infty)$ and $\lambda_j \in [0, g)$ for each $j = 1, ..., J$.

b)$^{\mathbf{A}}$ Provide some intuitions for the law of motion above.

c)$^{\mathbf{B}}$ Define the measure of country j's distance to the world technology frontier as

$$a_j(t) = \frac{A_j(t)}{A(t)}.$$

Show that

$$\dot{a}_j(t) = \sigma_j - \left(\sigma_j + g - \lambda_j\right) a_j(t). \tag{4}$$

d)$^{\mathbf{B}}$ The world's equilibrium is the sequence $\left\{\left[k_j(t), a_j(t)\right]_{t=0}^{+\infty}\right\}_{j=1}^{J}$ such that the law of motion in **a)** as well as equation (4) are both satisfied. A steady-state world equilibrium is then defined as a steady state of this equilibrium path, that is, an equilibrium with $\dot{k}_j(t) = \dot{a}_j(t) = 0$ for each $j = 1, ..., J$. Show that there exists a unique, globally stable steady-state world equilibrium in which income per capita in all countries grows at the same rate $g > 0$. Moreover, for each $j = 1, ..., J$, compute the steady-state world equilibrium $\left\{k_j^*, a_j^*\right\}_{j=1}^{J}$. Does your result imply that all countries will converge to the same level of income per capita?

References

Acemoglu, D. (2009): *Introduction to Modern Economic Growth*. Princeton: Princeton University Press (*forthcoming*).

Barro, R. J. and X. Sala-í-Martin (2004): *Economic Growth (2nd Ed.)*. Cambridge: MIT Press.

Chiang, A. C. and K. Wainwright (2005): *Fundamental Methods of Mathematical Economics (4th Ed.)*. Boston: McGraw-Hill Irwin.

de la Fuente, A. (2000): *Mathematical Methods and Models for Economists*. New York: Cambridge University Press.

Hall, R. E. and C. I. Jones (1999): "Why Do Some Countries Produce So Much More Output Than Others?"*Quarterly Journal of Economics*, 114, February, 83–116.

Hogg, R. V., McKean, J. W. and A. T. Craig (2004): *Introduction to Mathematical Statistics (6th Ed.)*. New Jersey: Prentice-Hall.

Hoy, M., Livernois, J., McKenna C., Stengos, T. and R. Rees (2001): *Mathematics for Economics (2nd Ed.)*. Cambridge: MIT Press.

Kolmogorov, A. N. and S. V. Fomin (1970): *Introductory Real Analysis*. New Jersey: Prentice-Hall.

Mas-Colell, A., Whinston, M. D. and J. R. Green (1995): *Microeconomic Theory*. New York: Oxford University Press.

Romer, D. (2006): *Advanced Macroeconomics (3rd Ed.)*. Boston: McGraw-Hill Irwin.

Rubinstein, A. (2006): *Lecture Notes in Microeconomic Theory: The Economic Agent*. Princeton: Princeton University Press.

Rudin, W. (1976): *Principles of Mathematical Analysis (3rd Ed.)*. New York: McGraw-Hill.
Schlicht, E. (2006): "A Variant of Uzawa's Theorem."*Economics Bulletin* 5, December, 1–5.
Simon, C. P. and L. Blume (1994): *Mathematics for Economists*. New York: W. W. Norton & Company.
Solow, R. M. (1957): "Technical Change and the Aggregate Production Function."*Review of Economics and Statistics* 39, August, 312–320.
Solow, R. M. (2000): *Growth Theory: An Exposition (2nd Ed.)*. New York: Oxford University Press.
Strang, G. (1991): *Calculus*. Wellesley, MA: Wellesley-Cambridge Press.
Swan, T. W. (1956): "Economic Growth and Capital Accumulation."*Economic Record* 32, November, 334–361.
Sydsæter, K. and P. Hammond (2005): *Essential Mathematics for Economic Analysis (2nd Ed.)*. New Jersey: Prentice Hall International.
Sydsæter, K., Hammond, P., Seierstad, A. and A. Strøm (2005): *Further Mathematics for Economic Analysis*. New Jersey: Prentice Hall International.
Sydsæter, Strøm, A. and P. Berck (2005): *Economists' Mathematical Manual (4th Ed.)*. Heidelberg: Springer Verlag.
Varian, H. R. (1992): *Microeconomic Analysis (3rd Ed.)*. New York: W. W. Norton & Company.
Uzawa, H. (1961): "Neutral Inventions and the Stability of Growth Equilibrium."*Review of Economic Studies* 28, April, 117–124.

Appendix

A Useful Results of Mathematics

A.1 Homogenous Function

For any scalar r the real-valued function $F(x_1, x_2, \ldots, x_n) : \mathbb{R}^n \to \mathbb{R}$ is *homogeneous of degree r* if

$$F(\lambda x_1, \lambda x_2, \ldots, \lambda x_n) = \lambda^r F(x_1, x_2, \ldots, x_n), \forall x_1, x_2, \ldots, x_n \text{ and } \lambda > 0.$$

Homogenous function has the following properties:

Theorem A.1 *If $F(x_1, x_2, \ldots, x_n)$ is homogeneous of degree r then the partial derivative functions*

$$F_i := \frac{\partial F(\lambda x_1, \lambda x_2, \ldots, \lambda x_n)}{\partial x_i}, \forall i \in \{1, \ldots, n\}$$

are homogeneous of degree $r - 1$.

Proof Take an arbitrary $\lambda > 0$, then $\forall x_i$

$$\begin{aligned} F(\lambda x_1, \lambda x_2, \ldots, \lambda x_n) - \lambda^r F(x_1, x_2, \ldots, x_n) = 0 \\ \lambda \frac{\partial F(\lambda x_1, \lambda x_2, \ldots, \lambda x_n)}{\partial x_i} - \lambda^r \frac{\partial F(x_1, x_2, \ldots, x_n)}{\partial x_i} = 0. \end{aligned}$$

Put it in another way,

$$\frac{\partial F}{\partial x_i}(\lambda x_1, \lambda x_2, \ldots, \lambda x_n) = \lambda^{r-1} \frac{\partial F}{\partial x_i}(x_1, x_2, \ldots, x_n). \qquad \square$$

Theorem A.2 (Euler's Formula) *Suppose* $F(x_1, x_2, \ldots, x_n)$ *is homogeneous of degree* r *and differentiable. Then at any* $(x_1, x_2, \ldots, x_n)$

$$\sum_{i=1}^{n} \frac{\partial F(x_1, x_2, \ldots, x_n)}{\partial x_i} x_i = r F(x_1, x_2, \ldots, x_n).$$

Proof By definition for arbitrary $\lambda > 0$

$$F(\lambda x_1, \lambda x_2, \ldots, \lambda x_n) - \lambda^r F(x_1, x_2, \ldots, x_n) = 0.$$

Differentiate it with respect to λ

$$\sum_{i=1}^{n} \frac{\partial F(\lambda x_1, \lambda x_2, \ldots, \lambda x_n)}{\partial (\lambda x_i)} x_i = r \lambda^{r-1} F(x_1, x_2, \ldots, x_n).$$

Since λ is arbitrarily taken, the equation above surely holds when $\lambda = 1$. $\square$

A.2 First-Order Ordinary Differential Equations

A.2.1 Homogenous Equation

Equation of the following form is called *homogenous equation*

$$\dot{x} + Ax = 0$$

where A is a constant, since the only constant term is 0 on the right hand side.

To solve it, rearrange the equation as

$$\begin{aligned}\frac{\dot{x}}{x} &= -A,\\ \frac{d\ln x}{dt} &= -A,\end{aligned}$$

integrate with respect to t and get the general solution

$$x(t) = \exp(-At + c).$$

Suppose it's known that $x(0) = x_0$, then substitute for c and get the special solution

$$x(t) = x_0 \exp(-At).$$

A.2.2 Linear Differential Equation with Propagator

Equation of the following form is called *linear differential equation with propagator*

$$\dot{x} = A(t)x + B(t) \tag{A.1}$$

where both A and B are functions of time and $B(t)$ is an additional term without x. The equation is called *autonomous* when $B(t)$ is a constant, i.e. its dependence on time only shows up throught the terms concerning $x(t)$.

The solution method, variation of constants, was proposed by Lagrange (1736-1813). Start from solving homogenous problem

$$\begin{aligned}\dot{x} &= A(t)x,\\ \int d\ln x &= \int A(t)dt,\end{aligned}$$

that is

$$x(t) = C\exp\left(\int A(t)dt\right). \tag{A.2}$$

Then Lagrange's idea is that by introducing perturbation term $B(t)$ (called *propagator* in physics and engineering) the constant term C in (A.2) becomes time dependent, i.e. $C = C(t)$. Differentiating (A.2) with respect to t gives

$$\dot{x} = \dot{C} \exp\left(\int A(t)dt\right) + A(t)x. \tag{A.3}$$

Compare (A.3) with (A.1) and get

$$\dot{C} = B(t) \exp\left(-\int A(t)dt\right).$$

Now it's simple to solve for C

$$C = \int B(\tau) \exp\left(-\int A(\tau)d\tau\right) d\tau + c. \tag{A.4}$$

Equation (A.1)'s solution is characterized by (A.2) and (A.4). Constant c can be solved when $x(0)$ is known.

A.2.3 Bernoulli Equation

Equation of the following form is called *Bernoulli equation*

$$\dot{x} = A(t)x + B(t)x^{\alpha}.$$

To solve it, define

$$y = x^{1-\alpha}$$

and subsitute for x. Then one gets linear differential equation with propagator

$$\dot{y} = (1-\alpha)A(t)y + (1-\alpha)B(t).$$

A.2.4 Example

Suppose that a representative infinitely living agent from a infinitely living household facing the following problem [6] :

- She has an initial level of assets stock $a(0)$ when she is born at $t = 0$;
- She receives a wage income flow $w(t)$ for $t \geq 0$;

[6] Readers will learn better interpretations in the lecture. Please concentrate on two questions here: (1) how to write down a flow budget constraint (law of motion); (2) how to solve the budget constraint as an ordinary differential equation.

- She receives a income flow $r(t)a(t)$ from renting her assets for $t \geq 0$;
- She generates a consumption flow $c(t)$ for $t \geq 0$;
- The population of the household grows at a rate n, implying that her assets are vaporizing at the same rate.

Then it's quite straight-forward to see her *life-time* budget constraint

$$a(0) \geq -\int_0^{+\infty} \exp\left(-\int_0^t [(r(\tau) - n)]\, d\tau\right)[w(t) - c(t)]\, dt, \tag{A.5}$$

or perhaps it's more straight-forward by putting it in another way

$$\int_0^{+\infty} \exp\left(-\int_0^t [(r(\tau) - n)]\, d\tau\right) c(t)dt \leq a(0) + \int_0^{+\infty} \exp\left(-\int_0^t [(r(\tau) - n)]\, d\tau\right) w(t)dt$$

meaning that the present value (by discounting everything with the market discount rate r and the demographic discount rate, i.e. population growth, n) of her life-time consumption should not exceed the present value of her life-time wealth.

It's also straight-forward to see her *flow* budget constraint can be written as

$$\begin{aligned} \dot{a}(t) &\leq w(t) + r(t)a(t) - c(t) - na(t) \\ &= [r(t) - n]\, a(t) + w(t) - c(t). \end{aligned}$$

Note that there is no boundary constraint for $t = +\infty$ which is directly imposed on the flow budget constraint.

A very important question is to ask whether this is equivalent to (A.5) (To make life easier from now on we take equality for both constraints). Note that the flow budget constraint has exactly the form of a linear ordinary differential equation with propagator, which suggests that we may check the equivalence by solving this differential equation.

First solve the equation without the propagator $w(t) - c(t)$ (simply a homogenous equation)

$$\begin{aligned} \dot{a}(t) &= [r(t) - n]\, a(t), \\ \frac{d \ln a(t)}{dt} &= r(t) - n, \\ a(t) &= C \exp\left(\int_0^t [(r(\tau) - n)]\, d\tau\right). \end{aligned}$$

Then take account of the effect from the propagator by setting constant C as a function of t, $C(t)$, and take derivative with respect to t

$$\dot{a}(t) = \dot{C}(t) \exp\left(\int_0^t [(r(\tau) - n)]\, d\tau \right) + \underbrace{C(t) \exp\left(\int_0^t [(r(\tau) - n)]\, d\tau \right)}_{a(t)} [(r(t) - n)]$$

$$= \dot{C}(t) \exp\left(\int_0^t [(r(\tau) - n)]\, d\tau \right) + [(r(t) - n)]\, a(t).$$

Compare it with the original equation

$$\dot{a}(t) = [r(t) - n]\, a(t) + w(t) - c(t)$$

one can see that

$$\dot{C}(t) \exp\left(\int_0^t [(r(\tau) - n)]\, d\tau \right) = w(t) - c(t).$$

Solve for $C(t)$ by integrating both sides with respect to t

$$C(t) = c + \int_0^t \exp\left(- \int_0^s [(r(\tau) - n)]\, d\tau \right) [w(s) - c(s)]\, ds.$$

Insert it into our interim result

$$a(t) = C(t) \exp\left(\int_0^t [(r(\tau) - n)]\, d\tau \right)$$

and we obtain

$$a(t) = c \exp\left(\int_0^t [(r(\tau) - n)]\, d\tau \right) + \int_0^t \exp\left(\int_s^t [(r(\tau) - n)]\, d\tau \right) [w(s) - c(s)]\, ds.$$

By applying that $a(t = 0) = a(0)$ solve to determine the constant $c = a(0)$. Therefore

$$a(t) = a(0)\exp\left(\int_0^t [(r(\tau) - n)]\,d\tau\right) + \int_0^t \exp\left(\int_s^t [(r(\tau) - n)]\,d\tau\right)[w(s) - c(s)]\,ds.$$

In terms of $a(0)$, it can be written as

$$a(0) = a(t)\exp\left(-\int_0^t [(r(\tau) - n)]\,d\tau\right) - \int_0^t \exp\left(-\int_0^s [(r(\tau) - n)]\,d\tau\right)[w(s) - c(s)]\,ds.$$

Alas, it seems different from (A.5)! These two constraints are **NOT** equivalent!

Later you will know that for the problems like this, as a result of optimization, the *transversality condition* leads to

$$\lim_{t\to+\infty} a(t)\exp\left(-\int_0^t [(r(\tau) - n)]\,d\tau\right) = 0,$$

implying that

$$a(0) = -\int_0^{+\infty} \exp\left(-\int_0^t [(r(\tau) - n)]\,d\tau\right)[w(t) - c(t)]\,dt.$$

And this is exactly the life-time constraint (A.5). Now we learned our first lesson from this exercise: The optimality conditions from the optimal control theory (which you will learn in the next class) require the transversality condition, making the two budget constraints interchangeable. The good news is that in the exercise of solving optimization problems you are allowed to use the flow budget constraint instead of the life-time budget constraint.

In the later lectures you will hear the so-called *No-Ponzi-Game constraint* saying that

$$\lim_{t\to+\infty} a(t)\exp\left(-\int_0^t [(r(\tau) - n)]\,d\tau\right) \geq 0,$$

which shall be added as a constraint in the very beginning your optimization exercise to rule out some economically implausible paths, and the transversality condition says that such contraint is in fact binding in the optimum.

From the solution procedure it is pretty clear to see why such constraint has to be imposed. Hopefully this gives readers the necessary strict (unfortunately, less entertaining) mathematical reasoning beyond the anecdotes of the famous gambler. □

Implicit Function Theorem Implicit function is defined through equation $F(x_1, x_2, \ldots, x_n, u) = 0$, $x_1, x_2, \ldots, x_n, u \in \mathbb{R}$ and $u(x_1, x_2, \ldots, x_n) : \mathbb{R}^n \to \mathbb{R}$. Given

$$\frac{\partial F}{\partial u} \neq 0$$

then

$$\frac{\partial u}{\partial x_i} := u_{x_i} = -\frac{\frac{\partial F}{\partial x_i}}{\frac{\partial F}{\partial u}}, \forall i \in \{1, \ldots, n\}.$$

L'Hôpital's Rule Suppose that $f(x) : \mathbb{R} \to \mathbb{R}$ and $g(x) : \mathbb{R} \to \mathbb{R}$ are twice continuously differentiable in the neighborhood of x^* where

$$\lim_{x \to x^*} f(x) = \lim_{x \to x^*} g(x) = 0$$

or

$$\lim_{x \to x^*} f(x) = \lim_{x \to x^*} g(x) = +\infty.$$

Then

$$\lim_{x \to x^*} \frac{f(x)}{g(x)} = \lim_{x \to x^*} \frac{f'(x)}{g'(x)}.$$

Leibnitz's Rule Suppose that function $f(x)$ is defined as

$$f(x) = \int_{u_1(x)}^{u_2(x)} g(t, x) dt, \quad x \in [a, b].$$

Suppose that

- $g(t, x)$ and $\frac{\partial g}{\partial x}$ are continuous in both t and x for $t \in [u_1(x), u_2(x)]$ and $x \in [a, b]$, as well as
- $u_1(x)$ and $u_2(x)$ are continuous and differentiable on $x \in [a, b]$,

then

$$\frac{d}{dx} f(x) = \int_{u_1(x)}^{u_2(x)} \frac{\partial}{\partial x} g(t, x) dt + g\left(u_2(x), x\right) \frac{d}{dx} u_2(x) - g\left(u_1(x), x\right) \frac{d}{dx} u_1(x).$$

As special cases, one can easily see that

$$\frac{d}{dx} \int_{a}^{u(x)} g(t) dt = g\left(u(x)\right) \frac{d}{dx} u(x),$$

$$\frac{d}{dx}\int_{u(x)}^{a} g(t)dt = -g\left(u(x)\right)\frac{d}{dx}u(x),$$

$$\frac{d}{dx}\int_{a}^{x} g(t)dt = g(x).$$

Log-Linearization Suppose that functions $x_1(t)$, $x_2(t)$, $\ldots$, $x_n(t)$ are all functions of t. Suppose that function $f(t)$ is the product of $x_i(t)$'s

$$f(t) = x_1(t)x_2(t)\ldots x_n(t),$$

then if we take logarithm to this equation

$$\ln f(t) = \ln x_1(t) + \ln x_2(t) + \ldots + \ln x_n(t)$$

and take derivative with respect to t, we get

$$\frac{\dot{f}(t)}{f(t)} = \frac{\dot{x}_1(t)}{x_1(t)} + \frac{\dot{x}_2(t)}{x_2(t)} + \ldots + \frac{\dot{x}_n(t)}{x_n(t)}$$

meaning that the change rate of $f(t)$ is the sum of the rates of $x_i(t)$'s.

Similarly, suppose that

$$g(t) = \frac{x_1(t)x_2(t)\ldots x_m(t)}{x_{m+1}(t)x_{m+2}(t)\ldots x_n(t)}$$

in which $1 \leq m < n$. Then

$$\frac{\dot{g}(t)}{g(t)} = \frac{\dot{x}_1(t)}{x_1(t)} + \frac{\dot{x}_2(t)}{x_2(t)} + \ldots + \frac{\dot{x}_m(t)}{x_m(t)} - \frac{\dot{x}_{m+1}(t)}{x_{m+1}(t)} - \frac{\dot{x}_{m+2}(t)}{x_{m+2}(t)} - \ldots - \frac{\dot{x}_n(t)}{x_n(t)}.$$

And we will see more techniques of log-linearization later.

Taylor Expansion If function $f(x) : \mathbb{R} \to \mathbb{R}$ is

- well defined on closed interval $[a, b]$ and
- continuously differentiable till $n + 1$-th order, i.e. $f'(x), f''(x), \ldots, f^{(n+1)}(x)$ exist for $x \in [a, b]$,

then

$$f(x) = \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(a)(x-a)^k + R_n(x),$$

in which

$$R_n(x) = \frac{1}{(n+1)!} f^{(n+1)}(\xi)(x-a)^{n+1}, (\xi \in (a, b)) \text{ (Lagrange residual)}$$

or

$$R_n(x) = \frac{1}{n!} f^{(n+1)} \left[a + \theta(x-a)\right](1-\theta)^n (x-a)^{n+1}, (\theta \in (0,1)) \text{ (Cauchy residual).}$$

If at (x_0, y_0) function $f(x, y) : \mathbb{R}^2 \to \mathbb{R}$ is continuously differentiable till $n + 1$-th order in the neighborhood $B(x_0, y_0)$, then for $(x, y) \in B(x_0, y_0)$

$$\begin{aligned} f(x,y) = &\sum_{k=0}^{n} \frac{1}{k!} \left[(x-x_0)\frac{\partial}{\partial x} + (y-y_0)\frac{\partial}{\partial y}\right]^k f(x_0, y_0) \\ &+ \frac{1}{(n+1)!}\left[(x-x_0)\frac{\partial}{\partial x} + (y-y_0)\frac{\partial}{\partial y}\right]^{n+1} f\left[x_0 + \theta(x-x_0), y_0 + \theta(y-y_0)\right] \end{aligned}$$

in which $\theta \in (0, 1)$, and the last term is the residual.

Besides these one can get similar equation for function $f(x_1, x_2, \ldots, x_n) : \mathbb{R}^n \to \mathbb{R}$ $(n > 2)$.

Example Taylor expansion for $f(x) = e^x$ around an arbitrary point $x^* = a$.

$$\begin{aligned} f(x) &= f(a) + f'(a)(x-a) + \frac{1}{2} f''(a)(x-a)^2 + \ldots + \frac{1}{n!} f^{(n)}(a)(x-a)^n + \ldots \\ &= f(a) + e^a(x-a) + \frac{1}{2} e^a (x-a)^2 + \ldots + \frac{1}{n!} e^a (x-a)^n + \ldots \end{aligned}$$

– this is the foundation of the log-linearization techniques which we will frequently use later. Especially when $x = 1$ and $a = 0$ the equation above collapses to

$$e = \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \ldots + \frac{1}{n!} + \ldots$$

– that's why people call $\ln := \log_e$ "natural logarithm".

Other very useful results can be derived here directly from the first-order approximation of e^x. Take $a = 0$, then

$$e^x = 1 + x + \frac{1}{2}x^2 + \ldots + \frac{1}{n!}x^n + \ldots$$

The first-order approximation gives

$$\begin{aligned} \lim_{x\to 0} e^x &= 1 + x, \\ \lim_{x\to 0} \ln(1+x) &= x. \end{aligned}$$

B Excercises

B.1 Ordinary Differential Equations

Solve the following problems concerning ordinary differential equations.

a) Find the general solution (and the particular solution for those with initial values given) for each of the following differential equations.

i)[A] $\dot{x} - 2x = 0$ with $x(0) = 3$. (*Answer:* $x(t) = 3e^{2t}$.)

ii)[A] $\dot{x} + 4x - 8 = 0$ with $x(0) = 2$. (*Answer:* $x(t) \equiv 2$.)

iii)[B] $\dot{x} + 2x - e^t = 0$ with $x(0) = \frac{3}{2}$. (*Answer:* $x(t) = \frac{1}{3}e^t + \frac{7}{6}e^{-2t}$.)

iv)[B] $\dot{x} = x - x^2$. (*Answer:* $x(t) = \frac{1}{1-c_2e^{-t}}$.)

v)[B] $\dot{x} + \frac{x}{t} = t^\alpha$. (*Answer:* $x(t) = \frac{1}{\alpha+2}t^{\alpha+1} + \frac{c}{t}$ *when* $\alpha \neq -2$*, and* $x(t) = \frac{\ln t}{t} + \frac{c}{t}$ *when* $\alpha = -2$.)

vi)[B] [7] $\dot{x} + \text{sign}(t)x = 0$ with $t \in (-\infty, +\infty)$ and $x(1) = 1$. (*Answer:* $x(t) = e^{1-|t|}$.)

b)[C] Show that if $\alpha > 0$ and $\lambda > 0$, then for any real β, every solution of

$$\frac{dy}{dx} + \alpha y(x) = \beta e^{-\lambda x}$$

satisfies $\lim_{x\to+\infty} y(x) = 0$. (The case $\alpha = \lambda$ requires special treatment.) Find the solution for $\beta = \lambda = 1$ which satisfies $y(0) = 1$. Sketch this solution for $0 \le x < +\infty$ for several values of α. In particular, show what happens when $\alpha \to 0$ and $\alpha \to +\infty$.

(*Answer:* $y = \frac{\beta}{\alpha-\lambda}e^{-\lambda x} + ce^{-\alpha x}$ *when* $\alpha \neq \lambda$, $y = (\beta x + c)e^{-\alpha x}$ *when* $\alpha = \lambda$*, and easy to see* $\lim_{x\to+\infty} y(x) = 0$. *When* $\beta = \lambda = 1$ *and* $y(0) = 1$, $y = \frac{1}{\alpha-1}e^{-x} + \frac{\alpha-2}{\alpha-1}e^{-\alpha x}$. *In the limits when* $\alpha \to 0$

$$\lim_{\alpha\to 0} y = 2 - e^{-x},$$

and when $\alpha \to +\infty$

$$\lim_{\alpha\to+\infty} y = \begin{cases} 1 \text{ if } x = 0, \\ 0 \text{ if } x > 0. \end{cases}$$

Plot $y(x)$ *for different* α*s*.)

---

[7] The sign function is defined as

$$\text{sign}(x) = \begin{cases} -1 & \text{for } x < 0 \\ 0 & \text{for } x = 0 \\ 1 & \text{for } x > 0 \end{cases}.$$

B.2 Solving Solow-Swan Model[C]

Suppose that the dynamic of the capital intensity $k(t)$ in an economy can be expressed as

$$\dot{k} = sAk^{\alpha} - (n + \delta)k.$$

As you can see the economy has a constant saving rate s, and the production follows the Cobb-Douglas technology $y = Ak^{\alpha}$. What's more, the capital intensity is eroded by the constant population growth rate n and depreciation rate δ.

Compute the steady-state capital intensity k^*. Suppose that the economy starts from $k(0) < k^*$. Compute the time path $k(t)$ and show that $\lim_{T\to+\infty} k(T) = k^*$. (Hint: This is a Bernoulli equation.)

(*Answer:* $k = \left\{\frac{sA}{n+\delta} + \left[k(0)^{1-\alpha} - \frac{sA}{n+\delta}\right] e^{-(1-\alpha)(n+\delta)t}\right\}^{\frac{1}{1-\alpha}}$. *Surely,* $\lim_{T\to+\infty} k(T) = k^*$.)